University of Rostock
Institute of Mathematics
PD Dr. M. Sawall
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Rostock, 11.04.2024

## 1. Tutorial on the lecture „Analysis and Numerics of Partial Differential Equations“

Problem 1.1:

Find all solutions of the following ordinary differential equations/systems:

a) $y'(t) = \frac{2}{t}y(t) + 4t^3$

b) $y'(t) = \dfrac{1 + (y(t))^2}{t}$

c) $x'(t) = y(t) + t, \quad y'(t) = x(t) - 1$

Problem 1.2:

Find all solutions $u = u(x, y)$ of the partial differential equation

$$12u_y - \frac{1}{2}u_{xy} = 0.$$

Problem 1.3:

Let $z = z(x, y)$ be an unknown function. Solve the following initial value problem:

$$2xz_x + yz_y = 0, \quad z(1, y) = y^2 + 5$$

Problem 1.4:

Let $\phi$ be twice differentiable and differentiable for all real $x$. Further be $c \in \mathbb{R}$.

Show that

$$u(x,t) = \frac{1}{2}(\phi(x + ct) + \phi(x - ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \phi(s)ds$$

is a solution of $u_{tt} = c^2 u_{xx}$.

Next show that this solution also satisfies the conditions $u(x, 0) = \phi(x); u_t(x, 0) = \phi(x)$ for all $x \in \mathbb{R}$.

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Rostock, 18.04.2024

## 2. Tutorial on the lecture „Analysis and Numerics of Partial Differential Equations“

Problem 2.1:
Let $f$ and $g$ be arbitrary differentiable functions. Find the second order partial differential equation which is independent of $f$ and $g$ and has the general solution

(a) $u(x, y) = f(x) + g(y)$,
(b) $u(x, y) = f(x) \cdot g(y)$.

Problem 2.2:
Consider the 2D vector field $\overrightarrow{v} = \big((x+1)y,\ x(y+1)\big)^T$.

(a) Compute the integral curves of $\overrightarrow{v}$.
(b) Determine the general solution $z(x, y)$ of $(x+1)yz_x + x(y+1)z_y = 0$.

Problem 2.3:
Solve the PDE

$$3yu_x - 2xu_y = 0$$

and find the particular solutions that satisfy the initial condition

(a) $u(x, y) = x^2$ on the line $y = x$ resp.
(b) $u(x, y) = 1 - x^2$ on the line $y = -x$.

Problem 2.4:
Compute the integral curves of the following vector fields and find the curves that pass through the point $P = (1, 1, 1)$

$$\text{(a)}\ \overrightarrow{v} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}, \qquad \text{(b)}\ \overrightarrow{v} = \begin{pmatrix} x \\ -y \\ y^2(1-z) \end{pmatrix}, \qquad \text{(c)}\ \overrightarrow{v} = \begin{pmatrix} y+z \\ z+x \\ x+y \end{pmatrix}.$$

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Rostock, 25.04.2024

## 3. Tutorial on the lecture „Analysis and Numerics of Partial Differential Equations“

Problem 3.1:

(a) Find the integral surfaces of $\overrightarrow{v} = (x^2,\ y^2,\ (x+y)z)^T$.

(b) Calculate the surface containing

$$\text{(i)}\ \ x = y,\ z = 1, \qquad \text{(ii)}\ \ x = y = z, \qquad \text{(iii)}\ \ x = 1,\ y = z.$$

Problem 3.2:

Calculate the integral curves with either the integral surfaces **or** using a system of ODEs to get the parametric form.

$$\text{(a)}\ \ \overrightarrow{v} = \begin{pmatrix} x \\ x+y \\ x+y+z \end{pmatrix}, \qquad \text{(b)}\ \ \overrightarrow{v} = \begin{pmatrix} x^2-1 \\ (y^2+1)(x+1) \\ xz+x-z-1 \end{pmatrix}.$$

Problem 3.3:

(a) Find the integral surfaces of the vector field

$$\overrightarrow{v} = \begin{pmatrix} (x+1)y \\ y^2+1 \\ y(z-1) \end{pmatrix}.$$

(b) Compute the solution $z = z(x, y)$ of the inhomogeneous PDE

$$(x+1)yz_x + (y^2+1)z_y = y(z-1).$$

(c) Further, solve the PDE

$$(x+1)yu_x + (y^2+1)u_y + y(z-1)u_z = 0$$

for $u = u(x, y, z)$.

Problem 3.4:

Find the solution $z = z(x, y)$ for the PDE

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = -1, \qquad 0 < y < x$$

with the initial condition $z(x, 0) = \ln x$ for $x > 0$.

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Rostock, 02.05.2024

## 4. Tutorial on the lecture „Analysis and Numerics of Partial Differential Equations“

Problem 4.1:

(a) Find the integral curves of $\overrightarrow{v} = (x, y, x + y + z)^T$

(b) Calculate the general solution of $xu_x + yu_y + (x + y + z)u_z = 0$, with $u = u(x, y, z)$.

(c) Solve $xz_x + yz_y = x + y + z$ with $z = z(x, y)$ unknown.

(d) Given is the initial value problem $xz_x + yz_y = x + y + z$, $z(x, x) = g(x)$. Under which conditions at $g(x)$ is it solvable in a neighborhood of $(x_0, x_0)$? Find all solutions.

Problem 4.2:
Classify the following PDEs

(a) $4u_{xx} + 4u_{xy} + u_{yy} = -4u$,

(b) $x^2u_{xx} + 2u_{xy} + y^2u_{yy} = 0$,

(c) $8u_{xx} + 6u_{yy} + 4u_{zz} + u_{xy} + 2u_{xz} + u_{yz} = 0$,

(d) $u_{xx}(x, y) + cu_{xy}(x, y) + u_{yy}(x, y) = 0$, with $c \in \mathbb{R}$,

(e)* $2u_{xy} - 2u_{xz} + 2u_{yz} + 3u_x - u = 0$.

Problem 4.3:
Classify the PDE

$$2u_{xx}(x, y) - u_{xy}(x, y) - u_{yy}(y, y) + u_y(x, y) - u(x, y) = 12\,.$$

What is the principal part of this PDE? Solve the characteristic equation and find the characteristic curves.

Problem 4.4:
Solve the characteristic equations for the following PDEs

(a) $u_{xx} - 4u_{xy} + 4u_{yy} + 2u_y + u = 0$

(b)* $u_{xx} + 2u_{xy} - 3u_{yy} + 3u_x - u = 0$

(c)* $\mathbf{e}^{2y}u_{xx} - \mathbf{e}^{2x}u_{yy} = 0$

Problem 4.5:
Transform $u_{xx} + 2u_{yy} + u_{zz} - 2u_{xz} = 0$ into the canonical. To this end compute a coordinate transformation

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

with a $3 \times 3$ matrix $A$.

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*Additional self-study exercise.

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Rostock, 16.05.2024

## 5. Tutorial on the lecture „Analysis and Numerics of Partial Differential Equations“

Problem 5.1:
Consider the PDE

$$21u_{xx} + 15u_{yy} + 18u_{zz} - 12u_{xz} + 12u_{yz} = 0.$$

(a) Classify the PDE.

(b) Transform it into canoncial form.

Problem 5.2:
Let $u = u(x, y)$ and $x, y \neq 0$ be given as well as the PDE

$$2x^2 u_{xx} + 5xyu_{xy} + 2y^2 u_{yy} + 8xu_x + 5yu_y = 0.$$

(a) Classify the PDE.

(b) Calculate the characteristic curves.

(c) Transform the PDE into canoncial form.

(d) Find the general solution of the PDE.

Problem 5.3:
Let $u = u(x, t)$ and $k \in \mathbb{R} \setminus \{0\}$ be given together with the PDE

$$u_{xx} + 2ku_{xt} + k^2 u_{tt} = 0.$$

(a) Find the characteristic curves of the PDE.

(b) Bring the PDE into canonical form.

(c) Find the solutions of the PDE.

Problem 5.4:

(a) Show that $u(x, y) = \ln(x^2 + y^2)$ is harmonic outside the origin.

(b) Show that $u(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ is harmonic outside the origin.

(c)* If $u$ and $v$ and $u^2 + v^2$ are harmonic, then $u$ and $v$ must be constant.

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*Additional self-study exercise.

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Rostock, 30.05.2024

## 6. Tutorial on the lecture „Analysis and Numerics of Partial Differential Equations“

Problem 6.1:
Let $\Omega = (-3,1) \times (-2,2)$ and $u : \mathbb{R}^2 \to \mathbb{R}$ be the solution of the Cauchy-problem

$$\Delta u = 0 \quad \text{for } x \in \Omega$$
$$u(x) = ax_1 - bx_2 \quad \text{for } x \in \partial\Omega \text{ and } a, b > 0$$

(a) Calculate the minimum and maximum of $u(x)$.

(b) Give the points where the minimum and maximum are reached.

Problem 6.2:
Show that for the coordinate tansformation $x = r\cos\varphi$ and $y = r\sin\varphi$ holds

(a) $\Delta u(x,y) = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi}$,

(b)* $u_y = u_r\frac{y}{r} + u_\varphi\frac{x}{r^2}$.

Problem 6.3:
Consider $\Omega$ to be a disc with Radius $R = 1$ and the center at the origin. Compute the solution of the boundary value problem

$$\Delta u = 0 \text{ in } \Omega, \qquad u(x,y) = x^2 - y^2 - x \text{ on } \partial\Omega.$$

Problem 6.4:
Solve the one-dimensional wave equation $u_{tt} - u_{xx} = g(x,t)$ on $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ for

(a) the homogeneous case $g(x,t) = 0$ and initial conditions $u(x,0) = \mathbf{e}^x$ and $u_t(x,0) = \sin(x)$,

(b) $g(x,t) = 0$ and initial conditions $u(x,0) = \ln(1+x^2)$ and $u_t(x,0) = x - 4$,

(c)* the inhomogeneous case $g(x,t) = \mathbf{e}^{at}$ and initial conditions $u(x,0) = u_t(x,0) = 0$.
Note: For the inhomogeneous wave equation $u_{tt} - u_{xx} = g(x,t)$ with $u(x,0) = u_t(x,0) = 0$ the solution is

$$u(x,t) = \frac{1}{2}\int_{s=0}^{t}\int_{y=x-(t-s)}^{x+(t-s)} g(y,s)\,\mathrm{d}y\,\mathrm{d}s.$$

Problem 6.5:
Solve the wave equation in the three-dimensional case with the initial conditions:

(a) $u(x,0) = 0,\ u_t(x,0) = x_2$

(b)* $u(x,0) = 0,\ u_t(x,0) = x_1x_2$

*Additional self-study exercise.

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Rostock, 06.06.2024

## 7. Tutorial on the lecture „Analysis and Numerics of Partial Differential Equations“

### Problem 7.1:

Compute the solution of the unbounded one-dimensional heat equation

$$u_t - u_{xx} = 0, \quad x \in \mathbb{R},\ t > 0$$

for the initial condition

(a) $u(x,0) = \mathbf{e}^{3x}$,

(b) $u(x,0) = 1$ if $x > 0$ and $\varphi(x) = 3$ if $x < 0$.

### Problem 7.2:

Find the solution of the following initial-boundary value problem:

$$\begin{aligned} &u_t - u_{xx} = 0, \text{ for } 0 < x < 1,\ t > 0 \\ &u(0,t) = u(1,t) = 0 \\ &u(x,0) = \sin^2(\pi x) \end{aligned}$$

### Problem 7.3:

Calculate numerical derivatives of

(a) $f(x) = \mathbf{e}^{x}$ at $x = 0$,

(b)* $f(x) = \ln(x)$ at $x = 1$,

(c)* $f(x) = \tan(x)$ at $x = \frac{\pi}{4}$

using forward, backward and central differences and the stepsize $h = 0.1$. Compare the results to the exact values.

### Problem 7.4:

Compute an approximation to $\Delta u$ for $u(x,y) = \sin(\pi x)\cos(\pi y)$ at $(x,y) = (0.4, 0.6)$ by the 5-point star using $\Delta x = \Delta y = h = 0.2$. Compare with the exact value.

*Additional self-study exercise.

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Rostock, 13.06.2024

8. Tutorial on the lecture „Analysis and Numerics of Partial Differential Equations“

Problem 8.1:
Calculate approximations to $\sqrt{2}$ by solving for $x \in [1,2]$ the Cauchy problem

$$y'(x) = \frac{1}{2y(x)}, \quad y(1) = 1$$

with the following methods:

(a) explicit Euler,

(b) implicit Euler,

(c) Euler-Heun.

Assume a stepsize $h = 0.2$ and compare the results with $\sqrt{2}$.

Problem 8.2:
Solve the boundary value problem

$$-u''(x) + vu'(x) = 1, \quad u(0) = 1, \quad u(1) = 2$$

(a) analytically,

(b) numerically, assuming the constant $v \in \{\pm 0.1, \pm 1, \pm 10, \pm 100\}$ and the stepsize $h = 0.1$. Use the finite difference method,

(c) try different methods for the first order term.

Problem 8.3:
Repeat Problem 8.2, but with the boundary condition $u'(1) = 0$ at the right boundary. The rest remains unchanged.

Problem 8.4:
Use the 5-point star and $h_x = h_y = 0.2$ to approximate the solution of

$$\begin{aligned} -\Delta u(x,y) &= 1 && \text{in } \Omega, \\ u(x,y) &= |x| + |y| && \text{on } \partial\Omega \end{aligned}$$

on $\Omega = (0,1)^2$.

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*Additional self-study exercise.

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Rostock, 20.06.2024

9. Tutorial on the lecture „Analysis and Numerics of Partial Differential Equations“

Problem 9.1:
Let $u_t + 3u_x = 0$ be given.

(a) Write down an upwind scheme for the PDE using backward differences in $x$-direction and forward differences in $t$-direction and discuss this scheme.

(b) Discuss the stability of the numerical solution with respect to the possible stepsizes.

Problem 9.2:
Consider again $u_t + 3u_x = 0$.

(a) Apply the Lax-Friedrich scheme and calculate $U_{2,1}$ to $U_{n-1,1}$, with

$$U_0 = u(0, x) = \max(0, 1 - |x|),$$

for $x \in [0, 2]$ and stepsizes $h = 0.4$, $k = 0.1$.

(b)* Apply the Lax-Wendroff scheme and calculate $U_{2,1}$ to $U_{n-1,1}$ as in (a).

(c) Compare the results to the analytical solution.

Problem 9.3:
Let $u_t + xu_x = 0$ be given.

(a) For which interval for $x$ is the applied Lax-Friedrich scheme stable with $k = h = 0.1$?

(b) For $x \in [0, 3]$ and $h = 0.1$ calculate the maximal time stepsize, for which Lax-Friedrich is still stable.

(c) Would the Lax-Wendroff scheme converge for this problem?

Problem 9.4*:
Solve the following initial value problem numerically

$$\begin{aligned} u_t + 0.2u_x &= -u, \qquad x \in \Omega,\ t > 0, \\ u(x, 0) &= \max(0, 1 + \cos(x)) \end{aligned}$$

(a) for $\Omega = [0, 2\pi]$ as well as

(b) for $\Omega = [\pi, 3\pi]$.

*Additional self-study exercise.

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Rostock, 27.06.2024

10. Tutorial on the lecture „Analysis and Numerics of Partial Differential Equations“

Problem 10.1:
Consider the boundary value problem

$$\begin{aligned} u_{xx} + 3u_{xy} - 7u_{yy} - u_y - u &= 3 && \text{in } \Omega, \\ u(x,y) &= g(x,y) && \text{on } \partial\Omega \end{aligned}$$

(a) Compute the difference stencil for the inner points with step sizes $h_x$ and $h_y$.

(b) Give the system of linear equations for $\Omega = (0,1)^2$, $h_x = h_y = 0.25$ and $g(x,y) = |x-y|$.

Problem 10.2:
Let the domain $\Omega = (0,2) \times (0,1) \cup (0,1) \times [1,2)$ and the boundary value problem

$$\begin{aligned} u_{xx} - u_{yy} + 4u_y - u_x &= 1 && \text{in } \Omega, \\ u(x,y) &= 5 && \text{on } \partial\Omega \end{aligned}$$

be given. Assume a uniform grid with step size $h_x = h_y = 0.2$.

(a) Sketch the domain and the grid and number the grid nodes.

(b) Write down the difference stencil.

(c) Determine the discretization matrix and the corresponding right-hand side.

(d) Compute and plot an approximation to the solution.

Problem 10.3:
Consider the domain

$$\Omega = \{(x,y) \in [0,1) \times (0,1) : \ x + y \leq 1\} \cup \{(x,y) \in (-1,0) \times (0,1) : \ x^2 + y^2 \leq 1\}.$$

(a) Sketch the area of $\Omega$.

(b)* How many boundary points are there for $h_x = h_y = 0.1$, when an elliptic PDE has to be solved with the 5 point star?

Problem 10.4*:
Solve the boundary value problem

$$\begin{aligned} -u_{xx}(x,y) + 3u_{yy}(x,y) &= 1 && \text{in } \Omega, \\ u(x,y) &= 2|x| + y && \text{on } \partial\Omega \end{aligned}$$

with $\Omega = (-1,1) \times (-1,1)$ and $h_x = h_y = 0.2$ for a uniform grid.

*Additional self-study exercise.

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Rostock, 04.07.2024

# 11. Tutorial on the lecture „Analysis and Numerics of Partial Differential Equations“

**Problem 11.1:**

Determine whether the following matrices are irreducible or not.

(a) $A = \begin{pmatrix} 1 & 7 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 9 & 0 & 0 & 5 \\ 0 & 3 & 0 & 8 \end{pmatrix}$ (b) $B = \begin{pmatrix} 3 & -3 & 5 & 9 \\ 0 & 1 & 0 & -2 \\ 9 & 3 & 7 & 2 \\ 0 & 7 & 0 & 6 \end{pmatrix}$

(c)* The discretization matrix of $\Delta u = 0$ for $\Omega = (0,1)^2$ with $h = 0.2$.

**Problem 11.2:**

Let be given the following parabolic PDE with initial and boundary conditions:

$$\begin{aligned} u_t - u_{xx} &= 3 \text{ in } (0,1) \times (0,20), \\ u(x,0) &= \begin{cases} 2x, & x \in (0,0.5], \\ 2(1-x), & x \in (0.5,1), \end{cases} \\ u(0,t) &= u(1,t) = 0. \end{aligned}$$

Calculate the first time layer with $h = 0.25$ and $k = 0.06$.

(a) Use the explicit Euler-method

(b) Use the Crank-Nicolson-method.

(c)* Use the implicit Euler-method.

**Problem 11.3:**

(a) Are the solutions of Problem 11.2 stable?

(b) What are the conditions that the solutions are stable for $h = 0.1$?

**Problem 11.4*:**

Let the following initial boundary value problem be given

$$\begin{aligned} u_t - u_{xx} &= 5 \text{ in } (-1,1) \times (0,10), \\ u(x,0) &= 10x^2, \text{ for } x \in (-1,1), \\ u(-1,t) &= u(1,t) = 10. \end{aligned}$$

Calculate the first time layer with $h = 0.25$ and $k = 0.01$.

*Additional self-study exercise.

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Rostock, 11.07.2024

## 12. Tutorial on the lecture „Analysis and Numerics of Partial Differential Equations“

**Problem 12.1:**
Let the following parabolic initial-boundary value problem be given

$$\begin{aligned} u_t(x,y,t) &= 2\Delta u(x,y,t) && \text{for } (x,y)\in\Omega=(-1,1)^2,\ t\in(0,10),\\ u(x,y,0) &= \max\big(0,(1-x^2)(1-y^2)\big) && \text{for } (x,y)\in\Omega,\\ u(x,y,t) &= 0 && \text{for } (x,y)\in\partial\Omega,\ t\in(0,10). \end{aligned}$$

Use the explicit Euler-method and set up the discretization matrix with $h_x = h_y = 0.25$ and $k = 0.03$. Calculate the approximations for the first time layer.

**Problem 12.2:**
Solve the following hyperbolic initial-boundary value problem

$$\begin{aligned} u_{tt}(x,t) &= 4u_{xx}(x,t) && \text{for } x\in\Omega=(0,1),\ t\in(0,10),\\ u(x,0) &= \exp\big(-40(x-0.5)^2\big) && \text{for } x\in\Omega,\\ u_t(x,0) &= 0 && \text{for } x\in\Omega,\\ u(0,t) &= u(1,t) = 0 \text{ for } x\in\partial\Omega,\ t\ge 0. \end{aligned}$$

Use finite differences with $h_x = 0.2$ and $k = 0.05$. Compare your approximation with the analytical solution.

**Problem 12.3:**
What can be said about the numerical stability of problem 12.2? What is the maximal stepsize in time direction regarding stability?

**Problem 12.4:**
Implement a code to approximate the solution of the initial-boundary value problem

$$\begin{aligned} u_{tt} - \Delta u &= 0 && \text{for } x\in\Omega=(-1,1)^2 \quad ,\\ u(x,y,0) &= \exp\big(-20(x^2+y^2)\big) && \text{for } x\in\Omega,\\ u_t(x,y,0) &= 0 && \text{for } x\in\Omega,\\ u(t,x) &= 0 && \text{for } x\in\partial\Omega,\ t>0. \end{aligned}$$

Use a grid with $N = 31$ nodes in each direction. Compute approximations for $t\in[0,5]$ with $k = 0.05$. Use a number of 100 eigen modes to approximate the initial displacement.

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$^*$Additional self-study exercise.

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Rostock, 18.07.2024

13. Tutorial on the lecture „Analysis and Numerics of Partial Differential Equations“

Problem 13.1:
Let the following parabolic initial-boundary value problem be given

$$\begin{aligned} &u_t - u_{xx} = 5 \text{ in } (-1,1)\times(0,10)\\ &u(x,0) = 10x^2, \text{ for } x \in (-1,1)\\ &u(-1,t) = u(1,t) = 10. \end{aligned}$$

Use the method of lines with stepsizes $h = 0.5$ and $k = 0.01$. To this end

(a) transform the PDE into an ODE system and

(b) use Heun's method to find the solution for the first layer.

Problem 13.2:
Give a weak formulation of $-\big((x^2+1)u(x)'\big)' + u(x) = x$ for $\Omega = (0,1)$ with $u(0) = u(1) = 0$.

Problem 13.3:
Derive a weak formulation of the boundary value problem

$$-xu_{xx} - u_{yy} + u_{xy} + 2u_x + u = 1 \quad \text{on } \Omega = (0,1)^2,\ u|_\Omega = 0.$$

Problem 13.4*:
Transform the hyperbolic initial-boundary value PDE

$$\begin{aligned} &u_{tt} = 4u_{xx}, && \text{for } x \in \Omega = (0,1),\ t > 0\\ &u(x,0) = \exp(-40(x-0.5)^2) && \text{for } x \in \bar{\Omega}\\ &u_t(x,0) = 0 && \text{for } x \in \bar{\Omega}\\ &u(0,t) = u(1,t) = 0 && \text{for } x \in \partial\Omega,\ t \geq 0 \end{aligned}$$

into an ODE system.

Problem 13.5*:
Show that the boundary value problem (clamped beam)

$$u^{(4)}(x) = f(x) \text{ in } \Omega, \qquad u(x) = u'(x) = 0 \text{ on } \partial\Omega$$

with $\Omega = (0,1)$ has the weak formulation

$$\int_0^1 u''(x)v''(x) = \int_0^1 f(x)v(x) \qquad \forall v \in \big\{v \in \mathcal{C}^2(\Omega)\cap\mathcal{C}(\bar{\Omega}) :\ v(0) = v(1) = v'(0) = v'(1) = 0\big\}.$$

*Additional self-study exercise.