# THE SIGN OF WREATH PRODUCT REPRESENTATIONS OF FINITE GROUPS

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Abstract. Let $G, H$ be finite groups. We asymptotically compute $|\mathrm{Hom}(G, H \wr A_n)|$, thereby establishing a conjecture of T. Müller.

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Let $G, H$ be finite groups. T. Müller[2] develloped an enumerative theory of homomorphisms $\varphi : G \to H \wr S_n$, as $n \to \infty$, and asked to generalize this theory to other sequences of groups. In particular, he conjectured the following.

**Conjecture 1.** *Let $G, H$ be a finite groups. Then we have for $n \to \infty$*

$$|\mathrm{Hom}(G, H \wr A_n)| = \Big(\frac{1}{1 + s_2(G)} + \mathcal{O}(e^{-cn^{1/|G|}})\Big)|\mathrm{Hom}(G, H \wr S_n)|,$$

*where $s_2(G)$ is the number of subgroups of index 2 in $G$.*

It is the aim of this note to proof this conjecture.

**Theorem 1.** *Conjecture 1 holds true for all finite groups $G$ and $H$.*

One of the applications of wreath product representations is the recognition of finite index subgroups of infinite groups. Let $\Gamma$ be an infinite group, $\Delta$ a subgroup of index $n$. The action of $\Gamma$ on the cosets $\Gamma/\Delta$ by shift defines a homomorphism $\varphi : \Gamma \to S_n$. If in addition we know the number of lifts of $\varphi$ to homomorphisms $\psi : \Gamma \to H \wr S_n$, we can compute $|\mathrm{Hom}(\Delta, H)|$. Doing so for different choices of $H$ one can in certain situations gather sufficient information to reconstruct $\Delta$. For $\Gamma$ being a free product of cyclic groups of prime order this reconstruction was completed in [3], for free products of arbitrary finite groups in [5].

Comparing homomorphisms into $H \wr A_n$ with homomorphisms into $H \wr S_n$ gives information on the embedding of finite index subgroups in large groups. More precisely define for a subgroup $\Delta$ of a group $\Gamma$ the core $\Delta^c$ as the normal subgroup $\bigcup_{\gamma\in\Gamma} \Delta^\gamma$. Then the case $H = 1$ of Theorem 1 implies that the probability that a random subgroup $\Delta$ of index $n$ of a free product $\Gamma = G_1 * \dots G_r$ of finite groups satisfies $\Gamma/\Delta^c \cong A_n$ converges to $\prod_{i=1}^r \frac{1}{1+s_2(G_i)}$. The case of general $H$ yields that the property $\Gamma/\Delta^c \cong A_n$ and the isomorphism type of $\Delta$ are asymptotically independent. It would be interesting to generalize such considerations to arbitrary virtually free groups.

We now turn to the proof of the Theorem. We denote by $\pi$ the canonical projection $H \wr S_n \to S_n$, and by $\epsilon$ the sign homomorphism $S_n \to C_2$. We view $C_2$ as $\{\pm 1\} \subseteq \mathbb{Z}$, that is, we write the group operation of $C_2$ multiplicatively, but allow for the addition of values as in $\mathbb{Z}$. Let $\varphi : G \to H \wr S_n$ be a homomorphism. Then $\epsilon \circ \pi \circ \varphi : G \to C_2$ has a kernel containing $G^2G'$. We denote the induced homomorphism $V = G/G^2G' \to C_2$ by $\overline{\varphi}$. To prove our theorem it is therefore sufficient to show that if $\varphi \in \mathrm{Hom}(G, H \wr S_n)$ is chosen at random, then the distribution of $\overline{\varphi}$ converges to a uniform distribution. This is certainly true if $s_2(G) = 0$, because then $G = G^2 G'$. We shall therefore from now on assume that $s_2(G) > 0$, that is, $V$ is a non-trivial elementary abelian 2-group. Then our claim is equivalent to the statement that for every non-trivial $v \in V$ we have

$$h_n^v(G,H) := \frac{1}{|\mathrm{Hom}(G, H \wr S_n)|} \sum_{\varphi \in \mathrm{Hom}(G, H\wr S_n)} \overline{\varphi}(v) \ll e^{-cn^{-1/|G|}},$$

where we identified $C_2$ with $\{\pm 1\} \subseteq \mathbb{Z}$.

We first compute the dependence of $h_n^v(G,H)$ on $H$.

**Lemma 1.** *Let $G, H$ be finite groups, $\varphi : G \to S_n$ a transitive permutation representation, $\pi : H \wr S_n \to S_n$ the canonical projection. Then the number of homomorphisms $\psi : G \to H \wr S_n$ satisfying $\pi \circ \psi = \varphi$ equals $|H|^{n-1}$.*

*Proof.* This follows form the proof of [3, Proposition 1], more precisely the equality between [3, (8)] and [3, (9)]. □

Next we compute the generating series of $h_n^v(G,H)$.

**Lemma 2.** *We have*

$$\sum_{\nu \geq 0} \frac{h_n^v(G,H)}{n!} x^n = \exp\left( \sum_{k=1}^{|G|} \sum_{\psi : G \to S_k\, transitive} \overline{\psi}(v) \frac{|H|^{k-1} x^k}{k!} \right).$$

*Proof.* This is a weighted version of the exponential principle, see e.g. [6, Theorem 5.1.4]. We only have to show that if $\pi \circ \varphi$ decomposes as $\pi \circ \varphi = \bigoplus a_i \psi_i$, where the $\psi_i$ are transitive permutation representations, then $\overline{\varphi}(v) = \prod \overline{\psi}(v)^{a_i}$. However, this follows immediately from the fact that $\epsilon$ is a homomorphism. □

To deal with the generating series we need a stability result similar to [4], note however, that here we do not require $P_2$ to be Hayman admissible. In fact it is easy to see that $\sum_{\nu \geq 0} \frac{h_n^v(G,H)}{n!} x^n$ is Hayman admissible if and only if $s_2(G) = 0$, which is precisely the case we are not interested in.

**Lemma 3.** *Let $P_1(x) = \sum_{\nu=1}^d a_\nu^{(1)} x^\nu$ be a polynomial with non-negative real coefficients, $a_d^{(1)} \neq 0$, and let $P_2 = \sum_{n=1}^d a_\nu^{(2)} x^\nu$ be a polynomial with complex coefficients satisfying $|a_\nu^{(2)}| \leq a_\nu^{(1)}$ for all $\nu \leq d$. Define the sequences $b_\nu^{(1)}, b_\nu^{(2)}$ by the relation $\sum_{\nu=0}^\infty \frac{b_\nu^{(i)}}{\nu!} x^\nu = e^{P_i(x)}$. Then either there exists some comples number $\zeta$ with $|\zeta| = 1$, such that $P_1(x) = P_2(\zeta x)$, or there is some $c > 0$ such that $|b_\nu^{(2)}| < e^{-c\nu^{1/d}} |b_\nu^{(1)}|$ for all $\nu$ sufficiently large.*

*Proof.* Let $\rho_n$ be the unique real solution of the equation $\rho P'(\rho) = n$. It then follows from Hayman's theorem [1, Theorem I] that

$$b_n^{(1)} \sim \frac{\exp(P_1(\rho_n))}{\rho_n^n \sqrt{2\pi(\rho_n P_1'(\rho_n) + \rho_n^2 P''(\rho_n))}}.$$

We now express $b_n^{(2)}$ using Cauchy's integral formula as

$$b_n^{(2)} = \frac{1}{2\pi i} \int\limits_{\partial B_{\rho_n}(0)} \frac{\exp(P_2(z))}{z^{n+1}}\, dz$$

to obtain

$$\begin{aligned}
|b_n^{(2)}| &\leq \frac{\max\limits_{|z|=\rho_n} |\exp(P_2(z)|}{\rho_n^n} \\
&\leq (\sqrt{2\pi}d + o(1))\rho_n^{d/2} \frac{\max\limits_{|z|=\rho_n} |\exp(P_2(z)|}{\exp(P_1(\rho_n))} b_n^{(1)} \\
&\ll n^{1/2} \frac{\max\limits_{|z|=\rho_n} |\exp(P_2(z)|}{\exp(P_1(\rho_n))} b_n^{(1)}
\end{aligned}$$

where we used the fact that for $n \to \infty$ we have $\rho_n \sim cn^{1/d}$. Hence it suffices to show that either there is some $\zeta$ with $P_1(x) = P_2(\zeta x)$, or

$$\max_{|z|=\rho_n} \Re P_2(z) < P_1(\rho_n) - cn^{1/d}$$

for some $c > 0$. If there exists some $\nu$ with $|a_\nu^{(2)}| < a_\nu^{(1)}$, then this follows immediately from the triangle inequality. Define the function $f : [0, 2\pi] \to [0, \infty)$ by

$$f(\theta) = \max_{\substack{1 \leq \nu \leq d \\ a_\nu \neq 0}} |\nu\theta + \arg a_\nu^{(2)} \bmod 2\pi|,$$

where we normalize mod in such a way that it takes values in $[-\pi, \pi)$. Being continuous, this function either has a zero $\xi$, or it is uniformly bounded from below by some positive constant $\delta$. In the first case we obtain $P_2(x) = P_1(e^{i\xi}x)$, while in the second we have

$$\begin{aligned}
P_1(\rho_n) - \Re P_2(e^{i\theta}\rho_n) &= \sum_{\nu=1}^{d} \big(1 - \Re e^{i(\nu\theta + \arg a_\nu^{(2)})}\big) a_\nu^{(1)} \rho_n^\nu \\
&\geq (1 - \Re e^{i\delta}) \min_{\substack{1 \leq \nu \leq d \\ a_\nu \neq 0}} a_\nu \rho_n^\nu \\
&\geq \big((1 - \cos\delta) \min_{\substack{1 \leq \nu \leq d \\ a_\nu \neq 0}} a_\nu\big) n^{1/d}.
\end{aligned}$$

Hence in either case our claim follows. $\square$

We can now finish the proof of the theorem. We have to show that for $v \neq 0$ we have $h_n^v(G, H) \ll e^{-cn^{1/d}} h_n^0(G, H)$. We have

$$\left| \sum_{\psi: G \to S_k \text{transitive}} \overline{\psi}(v) \frac{|H|^{k-1}}{k!} \right| \leq \sum_{\psi: G \to S_k \text{transitive}} \frac{|H|^{k-1}}{k!}$$

for every $k$, hence we can apply Lemma 3 to find that either our claim holds true, or there exists some $\zeta$ with $|\zeta| = 1$, such that

$$\sum_{k=1}^{|G|} \sum_{\psi: G \to S_k \text{transitive}} \overline{\psi}(v) \frac{|H|^{k-1} x^k}{k!} = \sum_{k=1}^{|G|} \sum_{\psi: G \to S_k \text{transitive}} \frac{|H|^{k-1} (\zeta x)^k}{k!}.$$

Consider first the coefficient of $x$ in these polynomials. There is only the trivial representation $G \to S_1 = 1$, hence the coefficient of $x$ on both sides equals 1, and we conclude $\zeta = 1$.

Next we consider the coefficient of $x^2$. Let $\overline{U} < G/G^2G'$ be a subspace of codimension 1, which does not contain $v$, and $U$ be the preimage of $\overline{U}$ under the canonical map $G \to G/G^2G'$. Then $\psi_0 : G \to G/U \cong S_2$ is a homomorphism, for which $\overline{\psi_0}(v) = -1$, and we conclude that

$$\Sigma_1 = \sum_{\psi:G\to S_2\text{transitive}} \overline{\psi}(v)\frac{|H|}{2} < \sum_{\psi:G\to S_2\text{transitive}} \frac{|H|}{2} = \Sigma_2,$$

say, while the equality $P_1(x) = P_2(\zeta x)$ implies that $\Sigma_1 = \zeta^2\Sigma_2$. Clearly both $\Sigma_1$ and $\Sigma_2$ are real, and we conclude that $\zeta^2 = -1$. However, this contradicts the condition $\zeta = 1$ obtained from the coefficient of $x$, and our claim follows.

## References


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