ASYMPTOTIC STABILITY FOR SETS OF POLYNOMIALS

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Abstract. We introduce the concept of asymptotic stability for a set of complex functions analytic around the origin, implicitly contained in an earlier paper of the first mentioned author ("Finite group actions and asymptotic expansion of $e^{P(z)}$", Combinatorica 17 (1997), 523 – 554). As a consequence of our main result we find that the collection of entire functions $\exp(\mathfrak{P})$ with $\mathfrak{P}$ the set of all real polynomials $P(z)$ satisfying Hayman's condition $[z^n]\exp(P(z)) > 0\,(n \geq n_0)$ is asymptotically stable. This answers a question raised in loc. cit.

1. Asymptotic stability

Let $\mathfrak{F}$ be a set of complex functions analytic in the origin, and for $f \in \mathfrak{F}$ let $f(z) = \sum_n \alpha_n^f z^n$ be the expansion of f around 0. $\mathfrak{F}$ is termed *asymptotically stable*, if

(i) $\forall\, f \in \mathfrak{F}\, \exists\, n_f \in \mathbb{N}_0\, \forall\, n \geq n_f:\ \alpha_n^f \neq 0$,

(ii) $\forall\, f, g \in \mathfrak{F}:\ \alpha_n^f \sim \alpha_n^g \to f = g$ in a neighbourhood of 0.

Here, for arithmetic functions f and g, the notation $f(n) \sim g(n)$ is short for

$$g(n) = f(n)\big(1 + o(1)\big), \quad n \to \infty.$$

A set of polynomials $\mathfrak{P} \subseteq \mathbb{C}[z]$ is called asymptotically stable, if the set of entire functions

$$\mathfrak{F} = \exp(\mathfrak{P}) := \big\{e^{P(z)} :\ P(z) \in \mathfrak{P}\big\}$$

is asymptotically stable. Define the degree of the zero polynomial to be -1. For a polynomial $P(z) = \sum_{\delta=0}^{d} c_\delta z^\delta$ of exact degree $d \geq 1$ with real coefficients c_δ consider the following two conditions:

($\mathcal{G}$) $\quad c_\delta = 0$ for $d/2 < \delta < d$,

($\mathcal{H}$) $\quad [z^n]e^{P(z)} > 0$ for all sufficiently large n.

Here, $[z^n]f(z)$ denotes the coefficient of z^n in the expansion of $f(z)$ around the origin. Asymptotically stable sets of functions first appeared in [3], where it was shown among other things that the set of polynomials

$$\mathfrak{P}_0 = \Big\{P(z) \in \mathbb{R}[z] :\ P(z) \text{ satisfies } (\mathcal{G}) \text{ and } (\mathcal{H})\Big\}$$

is asymptotically stable. Since for a finite group G we have[1]

$$\sum_{n=0}^{\infty} |\mathrm{Hom}(G, S_n)|\, z^n/n! = \exp\left(\sum_{\nu} |\{U : (G:U) = \nu\}|\, z^\nu/\nu\right),$$

asymptotic stability of $\mathfrak{P}_0$ implies in particular the following curious phenomenon ("asymptotic stability" of finite groups):

If for two finite groups G and H we have $|\mathrm{Hom}(G, S_n)| \sim |\mathrm{Hom}(H, S_n)|$ as $n \to \infty$, then these arithmetic functions must in fact coincide.

Condition $(\mathcal{H})$ arises in the work of Hayman [2], where it is shown that for a real polynomial $P(z)$ of degree at least 1 the function $e^{P(z)}$ is admissible in the complex plane in the sense of [2, pp. 68 - 69] if and only if $(\mathcal{H})$ holds; cf. [2, Theorem X]. The gap condition $(\mathcal{G})$ has turned out to be an efficient way of exploiting the fact that polynomials $P(z)$ arising from enumerative problems very often have the property that

$$\mathrm{supp}(P(z)) \subseteq \big\{\delta : \delta \mid \deg(P(z))\big\}.$$

In [3] the question was raised whether condition $(\mathcal{G})$ could be dropped while still maintaining asymptotic stability, i.e., whether the larger set of polynomials

$$\mathfrak{P} = \Big\{P(z) \in \mathbb{R}[z] :\ P(z) \text{ satisfies } (\mathcal{H})\Big\} \tag{1}$$

is asymptotically stable. The purpose of this note is to establish the following result, which in particular provides an affirmative answer to the latter question.

Theorem. *Let $P_1(z), P_2(z) \in \mathbb{R}[z]$ satisfy Hayman's condition $(\mathcal{H})$, for $i = 1, 2$ let $\{\alpha_n^{(i)}\}_{n\geq 0}$ be the coefficients of $e^{P_i(z)}$, and put $\Delta(z) := P_1(z) - P_2(z)$ as well as $m := \max\big(\deg(P_1(z)), \deg(P_2(z))\big)$.*

- (i) *Suppose that either $0 \leq \mu < m$, or $\mu = m$ and $\deg(P_1(z)) = \deg(P_2(z))$. Then we have $\deg(\Delta(z)) = \mu$ if and only if $|\log\alpha_n^{(1)} - \log\alpha_n^{(2)}| \asymp n^{\mu/m}$.*
- (ii) *If $\deg(P_1(z)) \neq \deg(P_2(z))$, then $|\log\alpha_n^{(1)} - \log\alpha_n^{(2)}| \asymp n\log n$.*

Here, $f(n) \asymp g(n)$ means that $f(n)$ and $g(n)$ are of the same order of magnitude; that is, there exist positive constants c_1, c_2 such that $c_1 f(n) \leq g(n) \leq c_2 f(n)$ for all n.

Corollary. *The set of polynomials $\mathfrak{P}$ defined in (1) is asymptotically stable.*

Proof. If $P_1(z), P_2(z) \in \mathbb{R}[z]$ are polynomials satisfying condition $(\mathcal{H})$ as well as $\alpha_n^{(1)} \sim \alpha_n^{(2)}$, then $\log\alpha_n^{(1)} - \log\alpha_n^{(2)} = o(1)$. By our theorem, $\deg(\Delta(z)) \notin [0, m]$, and hence $P_1(z) = P_2(z)$. □

[1] Cf. for instance [1, Prop. 1] or [4, Exercise 5.13].

2. Proof of the theorem

For $i = 1, 2$ put $P_i(z) = \sum_{\delta=0}^{d_i} c_\delta^{(i)} z^\delta$ with $c_{d_i}^{(i)} \neq 0$. Our assumptions that $P_1(z)$ and $P_2(z)$ have real coefficients and satisfy $(\mathcal{H})$ ensure via [2, Theorem X] that the functions $\exp(P_i(z))$ are admissible in the complex plane; in particular, in view of [2, formula (1.2)], we have $c_{d_i}^{(i)} > 0$. By [2, Theorem I] we find that, for $i = 1, 2$,

$$\alpha_n^{(i)} \sim \frac{\exp(P_i(\vartheta_n^{(i)}))}{\left(\vartheta_n^{(i)}\right)^n \sqrt{2\pi b_i(\vartheta_n^{(i)})}} \quad (n \to \infty),$$

where $\vartheta_n^{(i)}$ is the positive real root of the equation $\vartheta P_i'(\vartheta) = n$, and $b_i(\vartheta) = \vartheta P_i'(\vartheta) + \vartheta^2 P_i''(\vartheta)$. Since $c_{d_i}^{(i)} > 0$, the root $\vartheta_n^{(i)}$ is well defined and increasing for sufficiently large n, and unbounded as $n \to \infty$. This gives $\vartheta_n^{(i)} \sim \big(\frac{n}{d_i c_{d_i}^{(i)}}\big)^{1/d_i}$ and $b_i(\vartheta_n^{(i)}) \sim d_i\, n$, and hence

$$\alpha_n^{(i)} \sim \frac{\exp(P_i(\vartheta_n^{(i)}))}{\left(\vartheta_n^{(i)}\right)^n \sqrt{2\pi d_i n}} \quad (n \to \infty). \tag{2}$$

Formula (2) implies that

$$\log \alpha_n^{(1)} - \log \alpha_n^{(2)} = P_1(\vartheta_n^{(1)}) - P_2(\vartheta_n^{(2)}) - n\big(\log \vartheta_n^{(1)} - \log \vartheta_n^{(2)}\big) - \frac{1}{2}\big(\log d_1 - \log d_2\big) + o(1). \tag{3}$$

First consider case (ii), that is, the case when $d_1 \neq d_2$. Then, by (3),

$$\log \alpha_n^{(1)} - \log \alpha_n^{(2)} = \Big(\frac{1}{d_2} - \frac{1}{d_1}\Big) n \log n + \mathcal{O}(n),$$

that is,

$$\big|\log \alpha_n^{(1)} - \log \alpha_n^{(2)}\big| \asymp n \log n$$

as claimed.[2] Next suppose that $d_1 = d_2$. Then the right–hand side of (3) becomes

$$d_1^{-1} \log(c_{d_1}^{(1)}/c_{d_2}^{(2)})\, n + o(n);$$

in particular, we have $\deg(\Delta(z)) = m$ if and only if $|\log \alpha_n^{(1)} - \log \alpha_n^{(2)}| \asymp n$, which proves the last part of (i). Thirdly, for $m = 1$,

$$\log \alpha_n^{(1)} - \log \alpha_n^{(2)} = c_0^{(1)} - c_0^{(2)} + n \log(c_1^{(1)}/c_1^{(2)}) + o(1),$$

in particular, $\deg(\Delta(z)) = 0$ if and only if $|\log \alpha_n^{(1)} - \log \alpha_n^{(2)}| \asymp 1$. Hence, we may assume for the remainder of the argument that $m \geq 2$.

Now suppose that $0 \leq \mu := \deg(\Delta(z)) < m$. We want to show that in this case

[2]Here, as well as in certain other places below, a more precise estimate than the one stated is obtained, but not needed in the argument.

$|\log\alpha_n^{(1)} - \log\alpha_n^{(2)}| \asymp n^{\mu/m}$. We have

$$\begin{aligned} n - \vartheta_n^{(1)} P_2'(\vartheta_n^{(1)}) &= \vartheta_n^{(1)}\big[P_1'(\vartheta_n^{(1)}) - P_2'(\vartheta_n^{(1)})\big] \\ &= \vartheta_n^{(1)} \Delta'(\vartheta_n^{(1)}) \qquad (4) \\ &= a\mu\big(\vartheta_n^{(1)}\big)^{\mu} + o(n^{\mu/m}), \end{aligned}$$

where a is the leading coefficient of $\Delta(z)$, which we may suppose without loss of generality to be positive. Expanding $\vartheta P_2'(\vartheta)$ as Taylor series around $\vartheta_n^{(1)}$, we find that

$$\begin{aligned} \vartheta P_2'(\vartheta) - \vartheta_n^{(1)} P_2'(\vartheta_n^{(1)}) &= \Big(c_m^{(2)} m^2 \big(\vartheta_n^{(1)}\big)^{m-1} + \mathcal{O}\big(n^{\frac{m-2}{m}}\big)\Big)\big(\vartheta - \vartheta_n^{(1)}\big) \\ &\quad + \mathcal{O}\Big(n^{\frac{m-2}{m}}\big(\vartheta - \vartheta_n^{(1)}\big)^2 + \big(\vartheta - \vartheta_n^{(1)}\big)^m\Big). \qquad (5) \end{aligned}$$

If ϑ runs through the interval

$$I = \left[\vartheta_n^{(1)} - \frac{2a\mu}{m^2\, c_m^{(1)}},\ \vartheta_n^{(1)} + \frac{2a\mu}{m^2\, c_m^{(1)}}\right],$$

the right–hand side of (5) covers a range containing the interval

$$\Big[-(2-\varepsilon)a\mu\big(\vartheta_n^{(1)}\big)^{m-1},\ (2-\varepsilon)a\mu\big(\vartheta_n^{(1)}\big)^{m-1}\Big]$$

for every given $\varepsilon > 0$ and sufficiently large n depending on ε. Combining this observation with (4), we find that $n - \vartheta P_2'(\vartheta)$ changes sign in I, that is, $\vartheta_n^{(2)} \in I$ for large n; in particular we have $\vartheta_n^{(2)} - \vartheta_n^{(1)} = \mathcal{O}(1)$. Since $m \geq 2$, setting $\vartheta = \vartheta_n^{(2)}$ in (5) and rewriting the left–hand side via (4) now gives

$$a\mu\big(\vartheta_n^{(1)}\big)^{\mu} = \Big(c_m^{(1)} m^2 \big(\vartheta_n^{(1)}\big)^{m-1} + \mathcal{O}\big(n^{\frac{m-2}{m}}\big)\Big)\big(\vartheta_n^{(2)} - \vartheta_n^{(1)}\big) + o(n^{\mu/m}). \qquad (6)$$

For x, y real, $x \to \infty$, and $x - y = \mathcal{O}(1)$,

$$P_2(x) - P_2(y) = (x-y)\, P_2'(x) + \mathcal{O}\big((x-y)\, x^{m-2}\big).$$

Hence, applying (6), we have as $n \to \infty$

$$\begin{aligned} P_1(\vartheta_n^{(1)}) - P_2(\vartheta_n^{(2)}) &= \Delta(\vartheta_n^{(1)}) + P_2(\vartheta_n^{(1)}) - P_2(\vartheta_n^{(2)}) \\ &= \Delta(\vartheta_n^{(1)}) + (\vartheta_n^{(1)} - \vartheta_n^{(2)}) P_2'(\vartheta_n^{(1)}) + \mathcal{O}\big((\vartheta_n^{(1)} - \vartheta_n^{(2)})(\vartheta_n^{(1)})^{m-2}\big) \\ &= a\big(1 - \frac{\mu}{m}\big)\Big(\frac{n}{m\, c_m^{(1)}}\Big)^{\mu/m} + o(n^{\mu/m}). \end{aligned}$$

Moreover, using (6) again,

$$\begin{aligned} \log\vartheta_n^{(2)} - \log\vartheta_n^{(1)} &= \log\Big(1 + \frac{\vartheta_n^{(2)} - \vartheta_n^{(1)}}{\vartheta_n^{(1)}}\Big) \\ &= \frac{\vartheta_n^{(2)} - \vartheta_n^{(1)}}{\vartheta_n^{(1)}} + o\Big(\frac{\vartheta_n^{(2)} - \vartheta_n^{(1)}}{\vartheta_n^{(1)}}\Big) \\ &= \frac{a\mu}{m} n^{-1}\Big(\frac{n}{m\, c_m^{(1)}}\Big)^{\mu/m} + o\big(n^{\frac{\mu-m}{m}}\big). \end{aligned}$$

Inserting these estimates in (3) now yields

$$\log \alpha_n^{(1)} - \log \alpha_n^{(2)} = a\big(1 - \frac{\mu}{m}\big)\Big(\frac{n}{m\, c_m^{(1)}}\Big)^{\mu/m} + \frac{a\mu}{m}\Big(\frac{n}{m\, c_m^{(1)}}\Big)^{\mu/m} + o(n^{\mu/m})$$

$$= a\Big(\frac{n}{m\, c_m^{(1)}}\Big)^{\mu/m} + o(n^{\mu/m}) \asymp n^{\mu/m},$$

and our theorem is proven.

References

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[4] R. P. Stanley, *Enumerative Combinatorics* vol. 2, Cambridge University Press, 1999.

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