THE PRIMER NUMBER THEOREM FOR BEURLING'S GENERALIZED INTEGERS. NEW CASES

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ABSTRACT. This paper provides new cases of the prime number theorem for Beurling's generalized integers. Let N be the distribution of a generalized number system and let π be the distribution of its primes. It is shown that $N(x) = ax + O(x/\log^\gamma x)$ (C), $\gamma > 3/2$, where (C) stands for the *Cesàro sense*, is suffcient for the prime number theorem to hold, $\pi(x) \sim x/\log x$. The Cesàro asymptotic estimate explicitly means that

$$\int_1^x \frac{N(t) - at}{t} \left(1 - \frac{t}{x}\right)^k \mathrm{d}t = O\left(\frac{x}{\log^\gamma x}\right),$$

for some $k \in \mathbb{N}$. Therefore, it includes Beurling's classical condition. We also show that under these conditions the the Möbius function, associated the the generalized number system, has mean value equal to 0. The methods of this article are based on arguments from the theory of asymptotic behavior of Schwartz distributions and a complex tauberian theorem with local pseudo-function boundary behavior as the tauberian hypothesis.

1. INTRODUCTION

Let $1 < p_1 \le p_2, \ldots,$ be a non-decresing sequence of real numbers tending to infinity. Following Beurling [2], we shall call such a sequence $P = \{p_k\}_{k=1}^\infty$ a set of generalized prime numbers. We arrange the set of all possible products of generalized primes in a non-decreasing sequence $1 < n_1 \le n_2, \ldots,$ where every n_k is repeated as many times as it can be represented by $p_{k_1} p_{k_2} \ldots p_{k_m}$ with $k_j \le k_{j+1}$. The sequence $\{n_k\}_{k=1}^\infty$ is called the set of generalized integers.

The function π denotes the distribution of the generalized prime numbers,

$$(1.1) \qquad \pi(x) = \pi_P(x) = \sum_{p_k < x} 1 \,,$$

while the function N denotes the distribution of the generalized integers,

$$(1.2) \qquad N(x) = N_P(x) = \sum_{n_k < x} 1 \,.$$

Beurling's problem is then to find conditions over the function N which ensure the validity of the prime number theorem (PNT), i.e.,

$$(1.3) \qquad \pi(x) \sim \frac{x}{\log x} , \quad x \to \infty .$$

In his seminal work [2], Beurling proved that the condition

$$(1.4) \qquad N(x) = ax + O\left(\frac{x}{\log^\gamma x}\right) , \quad x \to \infty ,$$

where $a > 0$ and $\gamma > 3/2$, suffices for the PNT to hold. If $\gamma = 3/2$, then the PNT need not to hold, as showed first by Beurling by a continuous analog of a generalized prime number system, and then by Diamond [5] who exhibited an explicit example of generalized primes not satisfying the PNT in this case.

The present article studies new cases of the prime number theorem for Beurling's generalized primes. Our main goal is to show the following theorem.

Theorem 1. *Suppose there exist constants $a > 0$ and $\gamma > 3/2$ such that*

$$(1.5) \qquad N(x) = ax + O\left(\frac{x}{\log^\gamma x}\right) \quad \text{(C)} , \quad x \to \infty ,$$

Then the prime number theorem (1.3) holds.

In (1.5) the symbol (C) stands for the *Cesàro sense* [7]. It explcitly means that there exist some $m \in \mathbb{N}$ such that the following average estimate is satisfied:

$$(1.6) \qquad \int_1^x \frac{N(t) - at}{t} \left(1 - \frac{t}{x}\right)^m \mathrm{d}t = O\left(\frac{x}{\log^\gamma x}\right) , \quad x \to \infty .$$

We might have written (C, m) in (1.5) if (1.6) holds for a specific m; however, the value of m will be totally unimportant for our arguments and we therefore choose to omit it from the notation.

Naturally, if Beurling's condition (1.4) is verified, then (1.6) is automatically satified for all $m \in \mathbb{N}$. Thus, Theorem 1 is a natural extension of Beurling's theorem. Observe that our theorem is sharp, namely, the PNT does not necessary hold if $\gamma = 3/2$ in (1.5), as shown by Diamond counterexample itself.

It should be also noticed that Theorem 1 has a very general character in the following sense, it essentially allows terms with quite large oscillatory growth in the asymptotic estimate for $N(x) - ax$; for example, functions such as

$$Ax/\log^\gamma x + Bx^n \sin x + Cx^k e^{x^m} \cos(e^{x^m}), \dots,$$

may serve as an error term $O(x/\log^\gamma x)$ in the Cesàro sense.

In addition to the PNT, we also show that the Möbius function of a generalized number system has mean value equal to zero, provided that the condition (1.5) be satisfied.

2. Preliminaries and Notation

Throughout this article, the sequence $P = \{p_k\}_{k=1}^{\infty}$ stands for a fixed set of generalized prime numbers with generalized integers $\{n_k\}_{k=1}^{\infty}$. We shall always assume that the distribution of the generalized integers satisfies (1.5) for $\gamma > 1$. Let letter s always stands for a complex number $s = \sigma + it$.

2.1. Functions Related to Generalized Primes.
We denote by $\Lambda = \Lambda_P$ the *von Mangoldt function* of P, defined on the set of generalized integers as

$$(2.1) \qquad \Lambda(n_k) = \begin{cases} \log p_j\,, & \text{if } n_k = p_j^m\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

The *Chebyshev function* of P is defined as usual by

$$(2.2) \qquad \psi(x) = \psi_P(x) = \sum_{p_k^m < x} \log p_k = \sum_{n_k < x} \Lambda(n_k)\,.$$

Observe that (1.5) implies ([6, Lem. 3],[12],[20]) the ordinary asymptotic behavior

$$(2.3) \qquad N(x) \sim ax\,, \quad x \to \infty\,,$$

hence, the Dirichtlet series $\sum n_k^{-s}$ is easily seen to have abscissa of convergence less than 1. The *zeta function* of P is then the analytic function

$$(2.4) \qquad \zeta(s) = \zeta_P(s) = \sum_{k=1}^{\infty} \frac{1}{n_k^s}\,, \quad \Re e\, s > 1\,.$$

Becase of the well known result [1, Lem. 2E], the relation (1.1) is equivalent to the statement

$$(2.5) \qquad \psi(x) \sim x\,.$$

Our approach to the PNT (Theorem 1) will be to show (2.5).

The *Möbius function* of P is defined on the generalized integers by

$$(2.6) \quad \mu(n_k) = \mu_P(n_k) = \begin{cases} (-1)^m\,, & \text{if } n_k = p_{k_1} p_{k_2} \dots p_{k_m} \text{ with } k_j < k_{j+1}\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Finally, note [1, Lem. 2D] that we have

$$(2.7) \qquad \sum_{k=1}^{\infty} \frac{\Lambda(n_k)}{n_k^s} = -\frac{\zeta'(s)}{\zeta(s)}\,, \quad \Re e\, s > 1\,.$$

Likewise, one readily verifies the identity

$$(2.8) \qquad \sum_{k=1}^{\infty} \frac{\mu(n_k)}{n_k^s} = \frac{1}{\zeta(s)}\,, \quad \Re e\, s > 1\,.$$

2.2. Distributions and Generalized Asymptotics.

We shall make use of the theory of Schwartz distributions and some elements from asymptotic analysis on distribution spaces, the so called generalized asymptotics.

We denote by $\mathcal{D}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ the Schwartz spaces of the test functions consisting of smooth compactly supported functions and smooth rapidly decreasing functions, respectively, with their usual topologies. Their dual spaces, the spaces of distributions and tempered distributions, are denoted by $\mathcal{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$, respectively. The space $\mathcal{D}(0,\infty)$ is the space of smooth functions supported on the interval $(0,\infty)$, its dual is the space $\mathcal{D}'(0,\infty)$. The space $\mathcal{D}_{L^2}(\mathbb{R})$ is the space of smooth functions with all derivatives belonging to $L^2(\mathbb{R})$, its dual space is $\mathcal{D}'_{L^2}(\mathbb{R})$. The space $\mathcal{D}_{L^2}(\mathbb{R})$ is the intersection of all Sobolev spaces while $\mathcal{D}'_{L^2}(\mathbb{R})$ is the union of them. The space $\mathcal{D}'_{L^1}(\mathbb{R})$ is the dual space of $\dot{\mathcal{B}}(\mathbb{R})$, the space of smooth functions with all derivatives tending to 0 at $\pm\infty$. We refer to [17] for the very well known properties of these spaces.

We use the following Fourier transform

$$(2.9) \qquad \hat{\phi}(t) = \int_{-\infty}^{\infty} e^{-itx}\phi(x)\,\mathrm{d}x \ , \quad \text{for } \phi \in \mathcal{S}(\mathbb{R}) \ ;$$

it is defined by duality on $\mathcal{S}'(\mathbb{R})$, that is, if $f \in \mathcal{S}'(\mathbb{R})$ its Fourier transform is the tempered distribution given by

$$(2.10) \qquad \left\langle \hat{f}(t), \phi(t) \right\rangle = \left\langle f(x), \hat{\phi}(x) \right\rangle \ .$$

Let $f \in \mathcal{S}(\mathbb{R})$ be supported in $[0,\infty)$, its Laplace transform is the analytic function

$$(2.11) \qquad \mathcal{L}\{f; s\} = \mathcal{L}\{f(x); s\} = \left\langle f(x), e^{-sx} \right\rangle \ , \quad \Re e\, s > 0 \ .$$

The relation between the Laplace and Fourier transforms [3, 17] is given by $\hat{f}(t) = \lim_{\sigma\to 0^+} \mathcal{L}\{f; \sigma + it\}$, where the last limit is taken in the weak topology of $\mathcal{S}'(\mathbb{R})$.

We shall employ various standard tempered distributions. The Heaviside function is denoted by H, it is simply the characteristic function of $(0,\infty)$. The Dirac delta "function" δ is defined by $\langle \delta(x), \phi(x) \rangle = \phi(0)$, note that $H'(x) = \delta(x)$ (the derivative is understood in the distributional sense, of course). The Fourier transform of H is $\hat{H}(t) = -i/(t-i0)$, where the latter is defined as the distributional boundary value, on $\Re e\, s = 0$, of the analytic function $1/s$, $\Re e\, s > 0$, i.e.,

$$\left\langle \frac{-i}{t-i\sigma}, \phi(t) \right\rangle = \lim_{\sigma\to 0^+} \int_{-\infty}^{\infty} \frac{\phi(t)}{\sigma + it}\mathrm{d}t \ , \quad \phi \in \mathcal{S}(\mathbb{R}) \ .$$

We now turn our attention to asymptotic analysis of distributions [7, 16, 20], the so called generalized asymptotics.

Let $f \in \mathcal{D}'(\mathbb{R})$, a relation of the form

$$(2.12) \qquad \lim_{h\to\infty} f(x+h) = \beta \ , \quad \text{in } \mathcal{D}'(\mathbb{R}) \ ,$$

means that the limit is taken in the weak topology of $\mathcal{D}'(\mathbb{R})$, that is, for each test function from $\mathcal{D}(\mathbb{R})$ the following limit holds,

$$(2.13) \qquad \lim_{h\to\infty} \langle f(x+h), \phi(x)\rangle = \beta \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x \ .$$

The meaning of the expression $\lim_{h\to\infty} f(x+h) = \beta$ in $\mathcal{S}'(\mathbb{R})$ is clear. Relation (2.12) is an example of the so-called *S-asymptotics* of generalized functions (also called asymptotics by translation), we refer the reader to [16] for further properties of this concept.

On the other hand, we may attempt to study the asymptotic behavior of a distribution by looking at the behavior at large scale of the dilates $f(\lambda x)$ as $\lambda \to \infty$. In this case, we encounter the concept of *quasiasymptotic behavior* of distributions [7, 16, 19, 20]. We will study in connection to the PNT a particular case of this type of behavior, namely, a limit of the form

$$(2.14) \qquad \lim_{\lambda\to\infty} f(\lambda x) = \beta \ , \quad \text{in } \mathcal{D}'(\mathbb{R}) \ ,$$

Needless to say that (2.14) should be always interpreted in the weak topology of $\mathcal{D}'(\mathbb{R})$. We may also talk about (2.14) in other spaces of distributions with a clear meaning.

2.3. **Pseudo-functions.** A tempered distribution f is called a *pseudo-function* if $\hat{f} \in C_0(\mathbb{R})$, that is, $\hat{f}$ is a continuous function which vanishes at $\pm\infty$.

The distribution f is said to be *locally a pseudo-function* if it coincides with a psedo-function on each finite open interval. The property of being locally a pseudo-function admits a characterization [13] in terms of a generalized "Riemann-Lebesgue lemma"; indeed, f is locally a pseudo-function if and only if $e^{iht}f(t) = o(1)$ as $|h| \to \infty$ in the weak topology of $\mathcal{D}'(\mathbb{R})$, i.e., for each $\phi \in \mathcal{D}(\mathbb{R})$

$$(2.15) \qquad \lim_{|h|\to\infty} \left\langle f(t), e^{iht}\phi(t)\right\rangle = 0 \ .$$

We may call (2.15) the generalized Riemann-Lebesgue lemma for local pseudo-functions. It is then clear that if $f \in L^1_{\mathrm{loc}}$, then it is locally a pseudo-function, due to the classical Riemann-Lebesgue lemma.

Two important cases of local pseudo-functions will be of vital importance below. Let f be the Fourier transform of an element from $\mathcal{D}'_{L^1}(\mathbb{R})$, then f is locally a pseudo-function. It follows directly from the fact that Fourier transfoms of elements from $g \in \mathcal{D}'_{L^1}(\mathbb{R})$ are continuous functions [17, p. 256]. Let now f be the Fourier transform of a distribution from $\mathcal{D}'_{L^2}(\mathbb{R})$ and let $g \in L^2_{\mathrm{loc}}(\mathbb{R})$; because of the remark in [17, p. 256], the product $g \cdot f$ is a well defined distribution and it is locally a pseudo-function.

Let $G(s)$ be analytic on $\Re e\ s > \alpha$. We shall say that G *has local psedo-function boundary behavior* on the line $\Re e\ s = \alpha$ if it has distributional

boundary values [3] in such a line,

$$(2.16) \qquad \lim_{\sigma \to \alpha^+} \int_{-\infty}^{\infty} G(\sigma + it)\phi(t)\mathrm{d}t = \langle f(t), \phi(t) \rangle \ , \qquad \phi \in \mathcal{D}(\mathbb{R}) \ ,$$

and the boundary distribution f is locally a pseudo-function.

3. PROPERTIES OF THE ZETA FUNCTION

Our arguments for the proof of Theorem 1 rely on the properties of the zeta function. We shall derive such properties from those of the following special distribution. Define

$$(3.1) \qquad v(x) = \sum_{k=1}^{\infty} \frac{1}{n_k} \delta(x - \log n_k) \ .$$

Let us verify that v is a tempered distribution. We have that $g(x) = e^{-x}N(e^x)$ is a bounded function, hence $g \in \mathcal{S}'(\mathbb{R})$; therefore, $v = g' + g \in \mathcal{S}'(\mathbb{R})$.

The distribution v is intimately related to the zeta function. In fact, its Laplace transform is, $\Re e\, s > 0$,

$$(3.2) \qquad \mathcal{L}\{v, s\} = \langle v(x), e^{-sx} \rangle = \sum_{k=1}^{\infty} \frac{1}{n_k^{s+1}} = \zeta(s+1) \ .$$

Taking the boundary values of (3.2) on $\Re e\, s = 0$, in the distributional sense, we obtain the Fourier transform of v,

$$(3.3) \qquad \hat{v}(t) = \zeta(1 + it) \ .$$

Observe that we are interpreting (3.3) in the distributional sense and not as equality of functions, i.e., for each $\phi \in \mathcal{S}(\mathbb{R})$.

$$(3.4) \qquad \langle \hat{v}(t), \phi(t) \rangle = \lim_{\sigma \to 1^+} \int_{-\infty}^{\infty} \zeta(\sigma + it)\phi(t)\mathrm{d}t \ .$$

Next, we provide a lemma which establishes the main connection between (1.5) and the $S-$asymptotic properties of v.

Lemma 1. *The following assertions are equivalent:*

   (i) *In the sense of (1.6)*

$$(3.5) \qquad N(x) = ax + O\left(\frac{x}{\log^{\gamma} x}\right) \qquad (\mathrm{C}) \ , \qquad x \to \infty \ ,$$

   (ii) *There exist $m \in \mathbb{N}$ such that*

$$(3.6) \qquad \sum_{n_k < x} \left(1 - \frac{n_k}{x}\right)^m = \frac{ax}{m+1} + O\left(\frac{x}{\log^{\gamma} x}\right), \qquad x \to \infty \ ,$$

(iii) *In the sense of the quasiasymptotic behavior*

$$(3.7) \qquad N'(\lambda x) = \sum_{k=1}^{\infty} \delta(\lambda x - n_k) = aH(x) + O\left(\frac{1}{\log^{\gamma} \lambda}\right) ,$$

as $\lambda \to \infty$ *in the space* $\mathcal{D}'(\mathbb{R})$.

(iii) *In the sense of* $S-$*asymptotic behavior*

$$(3.8) \qquad v(x + h) = \sum_{k=1}^{\infty} \frac{1}{n_k} \delta(x + h - \log n_k) = a + O\left(\frac{1}{h^{\gamma}}\right) ,$$

as $h \to \infty$ *in the space* $\mathcal{S}'(\mathbb{R})$.

Proof. Later... □

Define the remainder distribution $E_1 = v - aH$. Because of Lemma 1, E_1 has the following $S-$asymptotic bound

$$(3.9) \qquad E_1(x + h) = O\left(\frac{1}{|h|^{\gamma}}\right) \quad \text{as } |h| \to \infty \ \text{ in } \mathcal{S}'(\mathbb{R}) .$$

For it, it is enough to argue $E_1(x + h) = v(x + h) - aH(x + h) = a(1 - H(x + h)) + v(x + h) - a = O(1/h^{\gamma})$, $h \to \infty$, in the space $\mathcal{S}'(\mathbb{R})$. On ther other hand, the estimate as $h \to -\infty$ follows easily from the fact that E_1 has support in $[0, \infty)$.

We now obtain the first properties of the zeta function. From the properties of E_1, we can show the continuity of $\zeta(s)$ on $\Re e\, s = 1$, $s \neq 1$.

Proposition 1. *Let N satisfy (3.5) for $\gamma > 1$. Then $\zeta(s) - a/(s-1)$ extends to a continuous function on $\Re e\, s \geq 1$, that is,*

$$(3.10) \qquad \zeta(1 + it) + \frac{ia}{t - i0} \in C(\mathbb{R}) .$$

Consequently, $t\zeta(1+it)$ is continuous over the whole real line and so $\zeta(1+it)$ is continuous in $\mathbb{R} \setminus \{0\}$.

Proof. Observe that $\hat{E}_1(t) = \zeta(1+it)+ia/(t-i0)$. Due to (3.9), $E_1 * \phi(h) = O(|h|^{-\gamma})$, $h \to \infty$, and so $E_1 * \phi \in L^1(\mathbb{R})$, for each $\phi \in \mathcal{S}(\mathbb{R})$. This is precisely Schwartz characterization [17, p. 201] of the space $\mathcal{D}'_{L^1}(\mathbb{R})$, and so $E_1 \in \mathcal{D}'_{L^1}(\mathbb{R})$. Therefore [17, p. 256], $\hat{E}_1$ is continuous. □

The ensuing lemma is the first step toward the non-vanishing property of ζ on $\Re e\, s = 1$, in the case $\gamma > 3/2$.

Lemma 2. *Let N satisfy (3.5) for $1 < \gamma < 2$. For each $t_0 \neq 0$ there exist $C = C_{t_0}$ such that for small $\sigma > 1$*

$$(3.11) \qquad |\zeta(\sigma + it_0) - \zeta(1 + it_0)| < C(\sigma - 1)^{\gamma - 1} .$$

Proof. Find $\varphi \in \mathcal{S}(\mathbb{R})$ such that $0 \notin \operatorname{supp} \hat{\varphi}$ and $\hat{\varphi}(t) = 1$ for t in a small neighborhood of t_0. Set $f = v * \varphi$, then $\hat{f}(t) = \hat{\varphi}(t)\hat{v}(t) = \hat{\varphi}(t)\zeta(1 + it)$. We have that f is a smooth function and it satisfies the estimate $f(x) = O(|x|^{-\gamma})$, in particular $f \in L^1(\mathbb{R})$ and so $\hat{f}$ is continuous.

Define the harmonic function

$$U(\sigma + it) = \left\langle f(x)H(x), e^{-itx}e^{-\sigma x} \right\rangle + \left\langle f(x)H(-x), e^{-itx}e^{\sigma x} \right\rangle \; ;$$

then U is a harmonic representation of $\hat{f}$ on $\Re e\, s > 0$, in the sense that $\lim_{\sigma \to 0^+} U(\sigma + it) = \hat{f}(t)$, uniformly over $\mathbb{R}$. We claim that $\zeta(\sigma + it) = U(\sigma - 1 + it) + O(\sigma - 1)$, $\sigma \to 1^+$. Consider $V(s) = \zeta(s + 1) - U(s)$, harmonic on $\Re e\, s > 0$. Because $\zeta(1 + it) - \hat{f}(t) = 0$ on a neighborhood of the point t_0, it follows that $V(s)$ converges uniformly to 0 in a neighborhood of t_0 as $\Re e\, s \to 0^+$. Then by applying the reflection principle [18, Section 3.4] to the real and imaginary parts of V, we have that V admits a harmonic extension to a (complex) neighborhood of t_0. Therefore, $U(s) - \zeta(s + 1) = V(s) = O(|s - t_0|)$, for $\Re e\, s > 0$ being sufficiently close to t_0. This shows the claim.

We now show $|U(\sigma - 1 + it_0) - \zeta(1 + t_0)| = O((\sigma - 1)^\gamma)$, $\sigma \to 1^+$; the estimate (3.11) inmediately follows from this claim. The estimate $f(x) = O(|x|^{-\gamma})$ and [7, Lem. 3.9.4, p. 153] imply the following quasiasymptotics

$$e^{-i\lambda t_0 x}f(\lambda x)H(x) = \mu_+ \frac{\delta(x)}{\lambda} + O\left(\frac{1}{\lambda^\gamma}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \; ,$$

and

$$e^{-i\lambda t_0 x}f(\lambda x)H(-x) = \mu_- \frac{\delta(x)}{\lambda} + O\left(\frac{1}{\lambda^\gamma}\right) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}) \; ,$$

where $\mu_\pm = \int_0^\infty f(\pm x)e^{\mp it_0 x}\mathrm{d}x$, and so $\mu_- + \mu_+ = \int_{-\infty}^\infty f(x)e^{it_0 x}\mathrm{d}x = \hat{f}(t_0) = \zeta(1 + t_0)$. Then, the two quasiasymptotics imply

$$\begin{aligned}
U\left(\frac{1}{\lambda} + it_0\right) &= \left\langle e^{-it_0 x}f(x)H(x), e^{-x/\lambda} \right\rangle + \left\langle e^{-it_0 x}f(x)H(-x), e^{x/\lambda} \right\rangle \\
&= \lambda\left\langle e^{-\lambda it_0 x}f(\lambda x)H(x), e^{-x} \right\rangle + \lambda\left\langle e^{-\lambda it_0 x}f(\lambda x)H(-x), e^x \right\rangle \\
&= \mu_+ \left\langle \delta(x), e^{-x} \right\rangle + \mu_- \left\langle \delta(x), e^x \right\rangle + O\left(\frac{1}{\lambda^{\gamma-1}}\right) \\
&= \mu_+ + \mu_- + O\left(\frac{1}{\lambda^{\gamma-1}}\right) = \zeta(1 + t_0) + O\left(\frac{1}{\lambda^{\gamma-1}}\right), \quad \lambda \to \infty \; .
\end{aligned}$$

Writing $\sigma - 1 = 1/\lambda$, the claim has been shown. This completes the proof of the lemma. $\qquad\qquad\square$

We are now in the position to show the non-vanishing of $\zeta(s)$ on $\Re e\, s = 1$, $s \neq 1$, for the case $\gamma > 3/2$. Actually, the proof is identically the same as the one of [1, Thrm. 8E], but we sketch it for the sake of completeness.

Theorem 2. *Let N satisfy (3.5) for $\gamma > 3/2$. Then, $t\zeta(1 + it) \neq 0$, for all $t \in \mathbb{R}$. Consequently, $1/((s - 1)\zeta(s))$ converges locally uniformly to a continuous function as $\Re e\, s \to 1^+$.*

Proof. The proof is essentially the classical argument of Hadmard. Without lost of generality we assume that $3/2 < \gamma < 2$. One uses the representation [1, Lem. 2C], which is also valid under our hypothesis,

$$\zeta(s) = \exp \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{p_k^{-s}}{j} \ ,$$

to conclude [1, Lem. 8B] that for any $m \in \mathbb{N}$ and $t_0 \in \mathbb{R}$

$$|\zeta(\sigma)|^{m+1} \, |\zeta(\sigma + it_0)|^{2m} \prod_{j=1}^{m} |\zeta(\sigma + i(j + 1)t_0)|^{2m-2j} \geq 1 \ .$$

If we now fix $t_0 \neq 0$ and m, Proposition 1 and the above inequality imply the existence of $A = A_{m,t_0} > 0$ such that for $0 < \sigma < 1$

$$1 \leq \frac{A \, |\zeta(\sigma + it_0)|^{2m}}{(\sigma - 1)^{m+1}} \ ,$$

or which is the same

$$D(\sigma - 1)^{1/2 + 1/(2m)} \leq |\zeta(\sigma + it_0)| \ ,$$

with $D = A^{1/2m}$.

Suppose we had $\zeta(1 + it_0) = 0$. Choose m such that $1/2 + 1/(2m) < \gamma - 1$. By the inequality (3.11) in Lemma 2, we would have

$$D(\sigma - 1)^{1/2 + 1/(2m)} \leq |\zeta(\sigma + it_0)| < C(\sigma - 1)^{\gamma - 1} \ ,$$

which is certainly absurd. Therefore, we must necessarily have $\zeta(1 + it) \neq 0$, for all $t \in \mathbb{R} \setminus \{0\}$. $\qquad \square$

We know obtain the boundary behavior of $-\zeta'(s)/\zeta(s) - 1/(s - 1)$.

Lemma 3. *Let N satisfy (3.5) for $\gamma > 3/2$. Then*

$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s - 1}$$

has local pseudo-function boundary behavior on the line $\Re e\, s = 1$.

Proof. We work with $s + 1$ instead of s and analyze the boundary behavior on $\Re e\, s = 0$. Recall E_1 was defined before (3.9). As observed in the proof of Proposition 1, we have that $\zeta(s + 1) - a/s = \mathcal{L}\{E_1; s\}$ converges uniformly over compacts to the continuous function $\hat{E}_1$, as $\Re e\, s \to 1^+$, so, by Theorem 2,

$$G_1(s) = \frac{1}{s\zeta(s + 1)} \left(\zeta(s + 1) - \frac{a}{s} \right) \ , \quad \Re e\, s > 0 \ ,$$

converges uniformly over finite intervals to a continuous function, and thus, its boundary value is a pseudofunction. Define $E_2(x) = (xE_1(x))'$. A quick computation shows that

$$\mathcal{L}\{E_2; s\} = s\zeta'(s+1) + \frac{a}{s} , \quad \Re e \, s > 0 .$$

Since $-\zeta'(s+1)/\zeta(s+1) - 1/s = -G_2(s) + G_1(s)$, where

$$G_2(s) = \frac{1}{s\zeta(s+1)} \cdot \mathcal{L}\{E_2; s\} ,$$

it is enough to see that $G_2(s)$ has local pseudo-function boundary behavior. Now, the $S-$asymptotic bound (3.9) implies that $E_2(x+h) = O(|h|^{-\gamma+1})$, and, because of the hypothesis $\gamma > 3/2$, we have that $E_2 * \phi \in L^2(\mathbb{R})$, for all $\phi \in \mathcal{S}(\mathbb{R})$. But this is presicely Schwartz's characterization [17, p. 201]of the distribution space $\mathcal{D}'_{L^2}(\mathbb{R})$; thus $E_2 \in \mathcal{D}'_{L^2}(\mathbb{R})$. As remarked in Section 2.3, the multiplication of the Fourier transform of elemets from $\mathcal{D}'_{L^2}(\mathbb{R})$ with elements of $L^2_{\text{loc}}(\mathbb{R})$ always gives rise to a distribution which is locally a pseudo-funtion. It remains to observe that $G_2(s)$ tends in $\mathcal{D}'(\mathbb{R})$ to $\hat{E}_2(t)/(t\zeta(1+it))$, which in view of the previous argument and the continuity of $1/(t\zeta(1+it))$ is locally a pseudo-function. $\qquad\square$

For future applications, we need a Chebyshev type estimate, it is the content of the next lemma.

Lemma 4. *Let N satisfy (3.5) for $\gamma > 3/2$. Then $\psi(x) = O(x)$, $x \to \infty$.*

Proof. Set $\tau(x) = e^{-x}\psi(e^x)$. The the crude estimate $\tau(x) \leq \log x e^{-x} N(x) = O(\log x)$, shows that $\tau \in \mathcal{S}'(\mathbb{R})$. Integration by parts in (2.7) shows that $\hat{\tau}(t) = G(1+it) + \hat{g}(t)$, where $G(1+it)$ is the distributional boundary value of the function

$$G(s) = \frac{1}{s}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right) ,$$

on the line $\Re e \, s = 1$, and g is the distribution given by

$$\hat{g}(t) = \frac{1}{1+it} \cdot \frac{-i}{t - i0} .$$

By Lemma 3, $G(1+it)$ is a pseudo-function. Since $\hat{H}(t) = -i/(t-i0)$ and the Fourier transform of $e^{-x}H(x)$ is precisely $1/(1+it)$, we obtain that g is the convolution of the latter two distributions, namely,

$$g(x) = H(x)\int_0^x e^{-(x-u)}du = H(x) - e^{-x}H(x) .$$

Pick $\phi \in \mathcal{S}'(\mathbb{R})$ such that it is non-negative and $\hat{\phi} \in \mathcal{D}'(\mathbb{R})$. Write, φ for the inverse Fourier transform of ϕ. Then,

$$\int_{-h}^{\infty} \tau(x+h)\phi(x)\mathrm{d}x = \langle \tau(x+h), \phi(x) \rangle$$

$$= \langle \tau(x+h) - g(x+h), \hat{\varphi}(x) \rangle + \int_{-h}^{\infty} g(x+h)\phi(x)\mathrm{d}x$$

$$= \left\langle G(1+it), e^{iht}\varphi(t) \right\rangle + O(1) = O(1) \ .$$

Notice that for x and h positive $e^x \tau(h) \leq \tau(x+h)$, which follows from the non-decreasing property of ψ. Finally, setting $C = \int_0^{\infty} e^{-x}\phi(x)\mathrm{d}x > 0$, we have

$$\tau(h) = C^{-1}\tau(h) \int_0^{\infty} e^{-x}\phi(x)\mathrm{d}x \leq C^{-1} \int_0^{\infty} \tau(x+h)\phi(x)\mathrm{d}x$$

$$\leq C^{-1} \int_{-h}^{\infty} \tau(x+h)\phi(x)\mathrm{d}x = O(1) \ .$$

$\square$

4. Tauberian Theorems

In this section we show Tauberian theorems from which we shall derive later the PNT and some properties of the Möbius function. Such theorems involve local pseudo-function behavior as the Tauberian hypothesis.

Theorem 3. *Let S be a non-decreasing function supported on $[0, \infty)$ and satisfying the growth condition $S(x) = O(e^x)$. Hence, the function*

(4.1)
$$\mathcal{L}\{\mathrm{d}S; s\} = \int_0^{\infty} e^{-sx}\mathrm{d}S(x)$$

is analytic on $\Re e\, s > 1$. If there exists a constant β such that the function

(4.2)
$$G(s) = \mathcal{L}\{\mathrm{d}S; s\} - \frac{\beta}{s-1}$$

has local pseudo-function boundary behavior on the line $\Re e\, s = 1$, then

(4.3)
$$S(x) \sim \beta e^x \ , \quad x \to \infty \ .$$

Proof. By subtracting $S(0)H(x)$, we may assume that $S(0) = 0$, so the derivative of S is given by the Stieltjes integral $\langle S'(x), \phi(x) \rangle = \int_0^{\infty} \phi(x)\mathrm{d}S(x)$. Let $M > 0$ such that $S(x) < Me^x$. Define $V(x) = e^{-x}S'(x)$.

We have that $e^{-x}S(x)$ is a bounded function, hence it is a tempered distribution and its set of translates is, in particular, weakly bounded; because differentiation is a continuous operator, the set of translates of $(e^{-x}S(x))'$ is weakly bounded, as well. Since $(e^{-x}S(x))' = -e^{-x}S(x) + V(x)$, we conclude that $V \in \mathcal{S}'(\mathbb{R})$ and $V(x+h) = O(1)$ in $\mathcal{S}'(\mathbb{R})$.

The Laplace transform of V on $\Re e\, s > 0$ is given by

$$\mathcal{L}\{V; s\} = \langle V(x), e^{-sx} \rangle = \int_0^\infty e^{-(s+1)x} \mathrm{d}S(x) = \mathcal{L}\{\mathrm{d}S; s+1\}\ .$$

Observe then that,

$$\hat{V}(t) + \frac{\beta i}{(t - i0)} = \lim_{\sigma \to 0^+} \mathcal{L}\{V(x) - \beta H(x); \sigma + it\}$$
$$= \lim_{\sigma \to 0^+} G(1 + \sigma + it)\ , \quad \text{in } \mathcal{D}'(\mathbb{R})\ .$$

Hence, by hypothesis, $\hat{V}(t) + i\beta/(t - i0)$ is locally a pseudo-function, therefore $e^{iht}(\hat{V}(t) + i\beta/(t - i0)) = o(1)$ as $h \to \infty$ in $\mathcal{D}'(\mathbb{R})$. Taking Fourier inverse transform, we conclude that $V(x + h) = H(x + h) + o(1) = \beta + o(1)$ as $h \to \infty$ in $\mathcal{F}(\mathcal{D}'(\mathbb{R}))$, the Fourier transform image of $\mathcal{D}'(\mathbb{R})$. Using the density of $\mathcal{F}(\mathcal{D}(\mathbb{R}))$ in $\mathcal{S}'(\mathbb{R})$ and the boundedness of $V(x + h)$, we conclude, by applying the Banach-Steinhaus theorem, that $V(x + h) = \beta + o(1)$ actually in $\mathcal{S}'(\mathbb{R})$. Multiplying by e^{x+h}, we obtain $S'(x + h) \sim e^{x+h}$ in $\mathcal{D}'(\mathbb{R})$.

Let $g(u) = S(\log u)$, then $\lim_{\lambda \to \infty} g'(\lambda u) = \beta$ in $\mathcal{D}'(0, \infty)$; indeed, let $\phi \in \mathcal{D}(0, \infty)$, then

$$\langle g'(\lambda u), \phi(u) \rangle = -\frac{1}{\lambda^2} \int_0^\infty S(\log u) \phi'\left(\frac{u}{\lambda}\right) \mathrm{d}u$$
$$= -\frac{1}{\lambda} \int_{-\infty}^\infty S(x + \log \lambda) e^x \phi'(e^x) \mathrm{d}x$$
$$= \frac{1}{\lambda} \langle S'(x + \log \lambda), \phi(e^x) \rangle$$
$$= \int_{-\infty}^\infty e^x \phi(e^x) \mathrm{d}x + o(1)$$
$$= \int_0^\infty \phi(u) \mathrm{d}u + o(1)\ , \quad \lambda \to \infty\ .$$

At this stage of the proof, we could apply first [19, Thrm. 4.1] and then [6, Lem. 3] (see also [20]) to g' and automatically conclude that $S(\log u) \sim \beta u$, which is equivalent to (4.3). Alternatively, we can proceed rather directly as follows. Let $\varepsilon > 0$ be an arbitrary small number; find ϕ_1 and $\phi_2 \in \mathcal{D}(0, \infty)$ with the following properties: $0 \leq \phi_i \leq 1$, $\mathrm{supp}\,\phi_1 \subseteq (0, 1]$, $\phi_1(u) = 1$ on $[\varepsilon, 1 - \varepsilon]$, $\mathrm{supp}\,\phi_2 \subseteq (0, 1 + \varepsilon]$, and finally, $\phi_2(u) = 1$ on $[\varepsilon, 1]$. Evaluating the quasiasymptotic limit of g' at ϕ_2, we obtain that

$$\limsup_{\lambda \to \infty} \frac{g(\lambda)}{\lambda} = \limsup_{\lambda \to \infty} \frac{1}{\lambda} \int_0^\lambda \mathrm{d}g(u) \leq \limsup_{\lambda \to \infty} \left(\frac{g(\varepsilon\lambda)}{\lambda} + \frac{1}{\lambda} \int_0^\infty \phi_2\left(\frac{u}{\lambda}\right) \mathrm{d}g(u) \right)$$
$$\leq M\varepsilon + \lim_{\lambda \to \infty} \langle g'(\lambda u), \phi_2(u) \rangle = M\varepsilon + \beta \int_0^\infty \phi_2(u) \mathrm{d}u \leq \beta + \varepsilon(M + \beta)\ .$$

Likewise, using now ϕ_1, we easily obtain that

$$\beta - 2\varepsilon\beta \leq \liminf_{\lambda \to \infty} \frac{g(\lambda)}{\lambda}\ .$$

Since ε was arbitrary, we conclude (4.3). $\qquad\qquad\square$

Theorem 3 implies the following Tauberian result for Dirichlet series.

Theorem 4. *Let $\{\lambda_k\}_{k=1}^\infty$ be a non-decreasing sequence of positive real numbers such that it tends to infinity and $\sum_{\lambda_k < x} 1 \sim ax$, for some non-negative a. Furthermore, assume that $\sum \lambda_k^{-s} - a/(s-1)$ has local pseudo-function boundary behavior on the line $\Re e\, s = 1$.*

Let $\{c_k\}_{k=1}^\infty$ be a sequence bounded from below, i.e., there exists $M > 0$ such that $c_k > -M$ for all k. Suppose that $\sum_{\lambda_k < x} c_k = O(x)$. Then

$$(4.4) \qquad\qquad \sum_{k=1}^\infty \frac{c_k}{\lambda_k^s}$$

it is analytic on $\Re e\, s > 1$. If there exists a constant β such that the distributional boundary value of

$$(4.5) \qquad\qquad G(s) = \sum_{k=1}^\infty \frac{c_k}{\lambda_k^s} - \frac{\beta}{s-1}$$

on the line $\Re e\, s = 1$ is locally a pseudo-function, then

$$(4.6) \qquad\qquad \sum_{\lambda_k < x} c_k \sim \beta x\ , \quad x \to \infty\ .$$

Proof. Set $S(x) = \sum_{\lambda_k < e^x}(c_k + M)$. Then $S(x) = O(e^x)$, and

$$\int_0^\infty e^{-st}\mathrm{d}S(t) = M\sum_{k=1}^\infty \frac{1}{\lambda_k^s} + \sum_{k=1}^\infty \frac{c_k}{\lambda_k^s}\ ;$$

Thus, S satisfies the hypotheses of Theorem 3, and so

$$S(x) \sim (\beta + aM)e^x,$$

from where (4.6) follows. $\qquad\qquad\square$

We emphasize that for $\lambda_k = n_k$ in Theorem 4 , generalized integers with N satisfying (3.5) for $\gamma > 1$, the hypothesis $\sum n_k^{-s} - a/(s-1)$ has local pseudo-function boundary behavior on $\Re e\, s = 1$ always holds, because of Proposition 1.

Theorem 4 generalizes Korevaar's Tauberian theorem from [13]. We remark that Korevaar's result was obtained via purely complex variable methods; here we use purely distributional methods! We remark that this result was used in [13] to conclude the classical Wiener-Ikehara theorem.

We need a variant of Theorem 4 with Tauberian hypothesis of *slow oscillation*. Recall a function τ is called slowly oscillating [12] if

$$(4.7) \qquad\qquad \lim_{h \to 0^+} \limsup_{x \to \infty} |\tau(x+h) - \tau(x)| = 0\ .$$

If (4.7) holds, then it is easy to see [12, p. 33] that $\tau(x) = O(x)$ and there exist $x_0, M > 0$ such that for $x \geq x_0$ and $h \geq 0$

$$(4.8) \qquad\qquad |\tau(x + h) - \tau(x)| \leq Ch .$$

Theorem 5. *Let $T \in L^1_{\mathrm{loc}}(\mathbb{R})$ be such that $\operatorname{supp} T \subseteq [0, \infty)$ and $\tau(x) = e^{-x}T(x)$ is slowly oscillating. Suppose there exists $\beta \in \mathbb{R}$ such that*

$$(4.9) \qquad\qquad G(s) = \mathcal{L}\{T; s\} - \frac{\beta}{s - 1}$$

has local pseudo-function boundary behavior on the line $\Re e\, s = 1$, then

$$(4.10) \qquad\qquad T(x) \sim \beta e^x , \quad x \to \infty .$$

Proof. Observe that since $\tau(x) = O(x)$, it is a tempered distribution and the Laplace transform of T is automatically well defined for $\Re e\, s > 1$. Since the Fourier transform of a compactly supported distributions is an entire function, we can assume that (4.8) holds in fact *for all $x \geq 0$ and $h \geq 0$*.

We first need to show that $e^{-x}T(x) = \tau(x)$ is bounded. For this, pick a test function $\eta \in \mathcal{D}(\mathbb{R})$ such that $\hat{\eta}(0) = 1/(2\pi)$ and set $\varphi = \hat{\eta}$. Next, for h large enough

$$\langle \tau(x + h), \varphi(x) \rangle = O(1) + \langle \tau(x) - H(x), \hat{\eta}(x - h) \rangle$$
$$= O(1) + \left\langle G(1 + it), \eta(t)e^{iht} \right\rangle = O(1) + o(1) = O(1) ,$$

because $G(1 + it)$ is a pseudo-function. Observe $\int_{-\infty}^{\infty} \varphi(x)\mathrm{d}x = 2\pi\eta(0) = 1$. Now, in view of (4.8),

$$|\tau(h)| \leq O(1) + \left| \int_{-h}^{\infty} (\tau(x + h) - \tau(h))\varphi(x)\mathrm{d}x \right| + |\tau(h)| \int_{-\infty}^{-h} |\varphi(x)| \, \mathrm{d}x$$
$$\leq O(1) + O(1) \int_{-\infty}^{\infty} |x\varphi(x)| \, \mathrm{d}x + O(h) \int_{-\infty}^{-h} |\varphi(x)| \, \mathrm{d}x = O(1) .$$

By adding a term of the form $Ke^x H(x)$, we may now assume $T \geq 0$. Define $S(x) = \int_0^x T(t)\mathrm{d}t$. The function S is increasing and, by $T(x) = O(e^x)$, has growth $S(x) = O(e^x)$. Furthermore,

$$\mathcal{L}\{S'; s\} - \frac{\beta}{s - 1} = \mathcal{L}\{T; s\} - \frac{\beta}{s - 1} ;$$

hence, by Theorem 2,

$$(4.11) \qquad\qquad \int_0^x T(t)\mathrm{d}t \sim \beta e^x .$$

In particular, the ordinary asymptotic behavior (4.11) implies the $S-$asymptotic behavior $\int_0^{x+h} T(t)\mathrm{d}t \sim \beta e^{x+h}$ as $h \to \infty$ in the space $\mathcal{D}'(\mathbb{R})$. Differentiating the latter and then dividing by e^{x+h}, we obtain

$$(4.12) \qquad\qquad \tau(x + h) = \beta + o(1) \quad \text{as } h \to \infty \text{ in } \mathcal{D}'(\mathbb{R}) .$$

The final step is to evaluate the $S-$asymptotic (4.12) at a suitable test function. Let $\varepsilon > 0$ be an arbitrary number. Choose $\phi \in \mathcal{D}(\mathbb{R})$ non-negative and supported in $[0, \varepsilon]$ such that $\int_0^\varepsilon \phi(x)\mathrm{d}x = 1$. Then,

$$
\begin{aligned}
\limsup_{h\to\infty} |\tau(h) - \beta| &\leq \limsup_{h\to\infty} \left| \beta - \int_0^\infty \tau(t)\phi(t-h)\mathrm{d}t \right| \\
&\quad + \limsup_{h\to\infty} \left| \int_0^\infty (\tau(t) - \tau(h))\phi(t-h)\mathrm{d}t \right| \\
&= \limsup_{h\to\infty} \left| \int_h^{h+\varepsilon} (\tau(t) - \tau(h))\phi(t-h)\mathrm{d}t \right| \\
&\leq \limsup_{h\to\infty} \sup_{t\in[h,h+\varepsilon]} |\tau(t) - \tau(h)| \,.
\end{aligned}
$$

Since ε was arbitrary, the slow oscillation (4.7) implies $\lim_{h\to\infty} \tau(h) = \beta$, which in turn is the same as (4.10). $\qquad\square$

5. Prime Number Theorem and Related Results for $\gamma > 3/2$

The prime number theorem, Theorem 1, follows now directly from our previous work. Indeed, it is enough to combine Lemma 3 and Lemma 4 with Theorem 4.

We end this article with a second application of the Tauberian theorems from Section 4. We now turn our attention to the Möbius function. We show its mean value is zero and $\sum \mu(n_k)/n_k = 0$, whenever $\gamma > 3/2$. Remarkably, it is well known that for ordinary prime numbers either of these conditions is equivalent to the PNT itself!

Theorem 6. *Let N satisfy (3.5) with $\gamma > 3/2$. Then,*

$$(5.1) \qquad \lim_{x\to\infty} \frac{1}{x} \sum_{n_k < x} \mu(n_k) = 0.$$

Proof. By using formula (2.8), Proposition 1, and Theorem 2, we have that

$$\sum_{k=1}^\infty \frac{\mu(n_k)}{n_k^s} = (s-1)\cdot \frac{1}{(s-1)\zeta(s)},$$

extends continuously to $\Re e\ s \geq 1$, and so this Dirichlet series has local pseudo-function boundary behavior on $\Re e\ s = 1$. Applying Theorem 4, we obtain (5.1) at once. $\qquad\square$

Theorem 7. *Let N satisfy (3.5) with $\gamma > 3/2$. Then,*

$$(5.2) \qquad \sum_{k=1}^\infty \frac{\mu(n_k)}{n_k} = 0\ .$$

Proof. Set $M(u) = \sum_{n_k < u} \mu(n_k)$ and

$$T(x) = e^x \int_0^x e^{-t} M(e^t)\mathrm{d}t\ .$$

Since $M(e^t) = o(e^t)$ (Theorem 6), we easily conclude that $e^{-x}T(x)$ is slowly oscillating. Notice that T is the convolution of $M(e^x)$ and $e^x H(x)$, then

$$\mathcal{L}\{T; s\} = \mathcal{L}\{M(e^x); s\}\,\mathcal{L}\{e^x H(x); s\}$$

$$= \frac{1}{s}\,\mathcal{L}\left\{\sum_{k=1}^{\infty}\mu_{n_k}\delta(x - \log n_k); s\right\}\int_0^{\infty} e^{-(s-1)x}\mathrm{d}x$$

$$= \frac{1}{s(s-1)}\sum_{k=1}^{\infty}\frac{\mu(n_k)}{n_k^s}$$

$$= \frac{1}{s(s-1)\zeta(s)}\ ,$$

from where it follows $\mathcal{L}\{T; s\}$ has local pseudo-function boundary behavior on $\Re e\, s = 1$. Theorem 5 yields $\lim_{x\to\infty}\int_0^x e^{-t}M(e^t)\mathrm{d}t = 0$; thus, a change of variables shows

$$\lim_{x\to\infty}\int_0^x \frac{M(u)}{u^2}\mathrm{d}u = 0\ .$$

We now derive (5.2) from the last limit and Theorem 6,

$$\sum_{n_k < x}\frac{\mu(n_k)}{n_k} = \int_0^x u^{-1}\mathrm{d}M(u) = \frac{M(x)}{x} + \int_0^x \frac{M(u)}{u^2}\mathrm{d}u = o(1)\ .$$

$$\square$$

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