SIFTED CHARACTER SUMS AND FREE QUOTIENTS OF BIANCHI GROUPS

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Abstract. We show that the Bianchi group $\mathrm{PSL}_2(\mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers in $\mathbb{Q}(\sqrt{d})$, $d<0$, has a free quotient of rank $\geq |d|^{\frac{1}{4}-\epsilon}$, as $|d|\to\infty$. To do so, we give an estimate for a sifted character sum.

Let $d<0$ be a fundamental discriminant, $\mathcal{O}_d$ be the ring of integers in $\mathbb{Q}(\sqrt{d})$, and $\Gamma_d=\mathrm{PSL}_2(\mathcal{O}_d)$ be the corresponding Bianchi group. Define the Zimmert set Z_d to be the set of all integers n satisfying the following conditions:

(1) $4n^2+3\leq |d|$, and $n\neq 2$;
(2) d is a quadratic non-residue modulo p for all odd prime factors p of n;
(3) If $d\not\equiv 5 \pmod 8$, then n is odd.

Denote by $r(d)$ the rank of the largest free quotient of Γ_d. R. Zimmert[5] proved that $r(d)\geq |Z_d|$. This relation was used by Mason, Odoni and Stothers[3] to show that for $|d|>10^{476}$, $r(d)\geq 2$, that is, Γ_d has a free non-abelian quotient, and that, as $|d|\to\infty$, we have $r(d)\gg\log d$. The difficulties in estimating $|Z_d|$ come from the fact that one has to bound sums of highly imprimitive characters, here, we avoid this problem by incorporating a sifting device into the character sum. This approach is similar to the one used in [4]. We will prove the following general estimate for imprimitive characters.

Theorem 1. *Let $\chi \pmod q$ be a character, P an integer. Then we have for each x and parameter $2\leq R\leq x$ the estimate*
$$\sum_{\substack{n\leq x\\ (n,P)=1}}\chi(n)\ll x^{1-\frac{1}{r}}R^{\frac{1}{r}}q^{\frac{r+1}{4r^2}+\epsilon}+x^{1+\epsilon}\sum_{\substack{R<t\leq x\\ t|P}}\frac{1}{t},$$
where $r\in\mathbb{N}$ can be chosen arbitrarily, if q is cubefree up to a factor 8, and $r\in\{1,2,3\}$ in general.

Corollary 1. *Let $d<0$ be a fundamental discriminant, $\mathcal{O}_d$ be the ring of integers in $\mathbb{Q}(\sqrt{d})$. Let $\Gamma_d=\mathrm{PSL}_2(\mathcal{O}_d)$ the Bianchi group corresponding to this ring. Then for each $\epsilon>0$ there is a d_0 such that for $|d|>d_0$ the group Γ_d maps onto a free group of rank $\geq |d|^{\frac{1}{4}-\epsilon}$.*

We will make use of the following version of Burgess' famous estimates for character sums.

Theorem 2. *Let $\chi \pmod q$ be a non-principal character. Then we have for each $x\geq 2$ the estimate*
$$\sum_{n\leq x}\chi(n)\ll x^{1-\frac{1}{r}}q^{\frac{r+1}{4r^2}+\epsilon},$$
where $r\in\mathbb{N}$ can be chosen arbitrarily, if q is cubefree, and $r\in\{1,2,3\}$ in general.

The restriction “q cubefree” can be somewhat lessened. If $q = q_1 q_2$ with q_2 cubefree, and $(q_1, q_2) = 1$, we can write each character $\chi \pmod q$ as $\chi_1\chi_2$ where χ_i is a character modulo q_i. Then we have

$$\sum_{n\leq x}\chi(n) = \sum_{(a,q)=1}\chi_1(a)\sum_{\substack{n\leq x\\ n\equiv a \bmod q_1}}\chi_2(n) \ll q_1(x/q_1)^{1-\frac{1}{r}} q_2^{\frac{r+1}{4r^2}+\epsilon}.$$

In particular, if q is cubefree up to a factor 8, we obtain up to a constant the same estimates as for cubefree integers.

To prove Theorem 1, fix a parameter R, and decompose the sum to be estimated as

$$\begin{aligned}\sum_{\substack{n\leq x\\ (n,P)=1}}\chi(n) &= \sum_{n\leq x}\chi(n)\Big(\sum_{t|(n,P)}\mu(t)\Big)\\ &= \sum_{n\leq x}\chi(n)\Big(\sum_{\substack{t\leq R\\ t|(n,P)}}\mu(t)\Big) - \sum_{\substack{n\leq x\\ (n,P)>1}}\chi(n)\Big(\sum_{\substack{t\leq R\\ t|(n,P)}}\mu(t)\Big)\\ &= \sum\nolimits_1 - \sum\nolimits_2,\end{aligned}$$

say. To estimate $\sum_1$, we interchange the order of summation and obtain

$$\sum\nolimits_1 \leq \sum_{t\leq R}\Big|\sum_{n\leq x/t}\chi(n)\Big| \ll \sum_{t\leq R}(x/t)^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}+\epsilon} \ll x^{1-\frac{1}{r}} R^{\frac{1}{r}} q^{\frac{r+1}{4r^2}+\epsilon},$$

for all r admissible by Theorem 2. To bound $\sum_2$, note that the inner sum vanishes for $2 \leq (n, P) \leq R$, hence, we may restrict the outer sum to terms with $(n, P) > R$. Trivially, the inner sum is bounded above by $\tau(n) \ll x^\epsilon$, and we obtain

$$\sum\nolimits_2 \ll x^\epsilon \#\{n \leq x : (n, P) > R\} \leq x^\epsilon \sum_{\substack{t>R\\ t|P}}\left[\frac{x}{t}\right].$$

The estimates for $\sum_1$ and $\sum_2$ now imply our theorem.

To deduce Corollary 1, fix $c < c' < \frac{1}{4}$, $0 < \epsilon < c' - c$, and let d be a sufficiently large integer. Since $\mathbb{Q}(\sqrt{-m^2 d}) = \mathbb{Q}(\sqrt{-d})$, we may suppose that d is squarefree. Denote by P be the product of all prime numbers p in the Zimmert set of d. Let χ be the character defined by $\chi(2) = 0$, $\chi(n) = \left(\frac{d}{n}\right)$ for n odd. Then χ is a character modulo $q = 4d$, and q is cubefree up to a possible factor 8. Setting $R = |d|^c$, $x = \frac{1}{2}\sqrt{|d| - 3}$, and $r = \left\lceil \frac{1}{1-4c'} \right\rceil$ in Theorem 1, we obtain

$$\begin{aligned}\sum_{\substack{n\leq x\\ (n,P)=1}}\chi(n) &\ll |d|^{\frac{1}{2}-(\frac{1}{4}-c')^2} + x^{1+\epsilon}\sum_{\substack{R<t\leq x\\ t|P}}\frac{1}{t}\\ &\ll x^{1-\delta} + x^{1-2c'+\epsilon}\#\{R < t \leq x : t|P\}\\ &\leq x^{1-\delta} + x^{1-c-c'}|Z_d|.\end{aligned}$$

On the other hand, if $n \leq x$ and $(n, Pd) = 1$, then $\chi(n) = 1$. Restricting the sum to prime values n we deduce

$$\sum_{\substack{n\leq x\\ (n,P)=1}}\chi(n) \geq \sum_{\substack{p\leq x\\ p\nmid P\\ p \text{ prim}}}\chi(p) \geq \pi(x) - |Z_d| - \omega(|d|),$$

and comparing these estimates we obtain $|Z_d| \gg |d|^c$.

References

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