# Lower Bounds on Covering Codes via Partition Matrices

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**Abstract**

Let $K_q(n,R)$ denote the minimal cardinality of a $q$-ary code of length $n$ and covering radius $R$. Let $\sigma_q(n,s;r)$ denote the minimal cardinality of a $q$-ary code of length $n$, which is $s$-surjective with radius $r$. In order to lower-bound $K_q(n,n-2)$ and $\sigma_q(n,s;s-2)$ we introduce partition matrices and their transversals. Our approach leads to a short new proof of a classical bound of Rodemich on $K_q(n,n-2)$ and to the new bound $K_q(n,n-2) \geq 3q-2n+2$, improving the first iff $5 \leq n < q \leq 2n-4$. We determine $K_q(q,q-2) = q-2+\sigma_2(q,2;0)$ if $q \leq 10$. Moreover, we obtain the new powerful recursive bound $K_{q+1}(n+1,R+1) \geq \min\{2(q+1), K_q(n,R)+1\}$.

**Keywords:** covering codes, surjective codes, lower bounds, partition matrices, transversals

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# 1 Introduction

In the whole paper let $q \geq 2$ and $\mathbb{Z}_q = \{0, 1, \ldots, q-1\}$. The following generalized surjective codes have been introduced by Kéri and Östergård.

**Definition 1** (Kéri, Östergård [6]). *Let $0 \leq r < s \leq n$. A $q$-ary code $C \subset \mathbb{Z}_q^n$ of length $n$ is called $s$-surjective with radius $r$ if for any $s$-tupel $(k_1, \ldots, k_s) \in \mathbb{Z}_n^s$ of pairwise distinct coordinates and any $s$-tuple $(x_1, \ldots, x_s) \in \mathbb{Z}_q^s$ there is a codeword $c = (c_0, \ldots, c_{n-1}) \in C$ such that $|\{i \in \{1, \ldots, s\} \mid c_{k_i} = x_i\}| \geq s - r$. Let $\sigma_q(n, s; r)$ denote the minimal cardinality of a $q$-ary code of length $n$, which is $s$-surjective with radius $r$.*

Clearly, $\sigma_q(n+1, s; r) \geq \sigma_q(n, s; r)$ and $\sigma_q(n, r+1; r) = q$. For bounds on $\sigma_q(n, s; r)$ and tables of $\sigma_q(n, s; 0)$ see Kéri, Östergård [6, 7, 8]. We make use of

**Theorem 2** (Kéri, Östergård [8]). $\sigma_3(5, 3; 1) = 7$,

a result obtained in [8] by computational means.

A $q$-ary code of length $n$ and covering radius (at most) $R$ is a code $C \subset \mathbb{Z}_q^n$, which is $n$-surjective with radius $R$. As usual we set $K_q(n, R) = \sigma_q(n, n; R)$. For a monograph on covering codes see [4]. An updated table of bounds on $K_q(n, R)$ is published online by Kéri [5]. The generalized surjective codes turned out to be a valuable tool in the theory of bounds for covering codes, see [6, 8].

In the present paper we introduce partition matrices and their transversals. The consideration of such matrices yields a natural, purely set combinatorial method to lower-bound $\sigma_q(n, s; s-2)$ and $K_q(n, n-2)$. Up to now the most powerful lower bound on $K_q(n, n-2)$ is due to Rodemich [12]:

$$K_q(n, n-2) \geq \frac{q^2}{n-1}. \quad (1)$$

We use our approach to give on the one hand a short new proof of this bound in a slightly improved version and on the other hand a substantial improvement in certain cases, see Theorem 7 and Theorem 8.

Of special interest is the case $n = q$. Recall the following result.

**Theorem 3** (Brace, Daykin [3], Kleitman, Spencer [9]). *$\sigma_2(n, 2; 0)$ equals the least integer $M$ satisfying*

$$n \leq \binom{M-1}{\lfloor \frac{M}{2} \rfloor - 1}.$$

Hence, $\sigma_2(2,2;0) = \sigma_2(3,2;0) = 4$, $\sigma_2(4,2;0) = 5$ and $\sigma_2(n,2;0) = 6$ if $5 \le n \le 10$. The bound

$$K_q(q, q-2) \le q - 2 + \sigma_2(q, 2; 0). \tag{2}$$

is a special case of [11, Theorem 8], also confer [4, Theorem 3.7.7]. It is an open problem, whether equality always holds. This is known to be the case for $q \le 4$, see [4, 5]. We extend equality to $q \le 10$, see Corollary 10.

This paper is organized as follows. Section 2 is fundamental since it presents the notion of a partition matrix and its connection to covering codes as well as the new powerful recursive bound $K_{q+1}(n+1, R+1) \ge \min\{2(q+1), K_q(n,R)+1\}$. Section 3 contains the announced improvements of Rodemich's bound (1), while Section 4 leads to six new exact values on $K_q(q, q-2)$. Section 5 collects all new lower bounds on $K_q(n,R)$ from this paper.

## 2 Partition Matrices and Covering Codes

The following definition is a modification of the one given in [1].

**Definition 4.** *A $q \times n$-matrix $\mathcal{P} = (P_{ik})$ $(i \in \mathbb{Z}_q, k \in \mathbb{Z}_n)$ of subsets of $\mathbb{Z}_M$ is called an $(n, M, q)$-partition matrix if the sets of every column of $\mathcal{P}$ form a partition of $\mathbb{Z}_M$. If additionally $|\bigcap_{k\in\mathbb{Z}_n} P_{i_k k}| \le 1$ for all words $(i_0, \dots, i_{n-1}) \in \mathbb{Z}_q^n$ then $\mathcal{P}$ is called strict.*

*A sequence of $s$ pairwise disjoint subsets from pairwise distinct columns of $\mathcal{P}$ is called an $s$-transversal (or a transversal of length $s$).*

**Theorem 5.** *If $2 \le s \le n$ then the following statements are equivalent:*

(i) *Every $(n, M, q)$-partition matrix has an $s$-transversal.*

(ii) *Every strict $(n, M, q)$-partition matrix has an $s$-transversal.*

(iii) *$\sigma_q(n, s; s-2) > M$.*

*Proof.* (i) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (iii): Let $C \subset \mathbb{Z}_q^n$ be a code of cardinality $M$. Let $\mathcal{C} = (c_{jk})$ $(j \in \mathbb{Z}_M, k \in \mathbb{Z}_n)$ be the $M \times n$-matrix obtained from $C$ by using the codewords row-wise in an arbitrary order. For $i \in \mathbb{Z}_q, k \in \mathbb{Z}_n$ set $P_{ik} = \{j \in \mathbb{Z}_M \mid c_{jk} = i\}$. Then $\mathcal{P} = (P_{ik})$ is a strict $(n, M, q)$-partition matrix. By assumption, it has an $s$-transversal $(P_{x_i k_i})_{i\in\{1,\dots,s\}}$. Then for every $j \in \mathbb{Z}_M$ the equation

$c_{jk_i} = x_i$ holds for at most one $i \in \{1, \ldots, s\}$. Hence, $C$ is not $s$-surjective with radius $s-2$.

(iii) $\Rightarrow$ (i): Let $\mathcal{P} = (P_{ik})$ be a $(n, M, q)$-partition matrix. For every $j \in \mathbb{Z}_M$ and every $k \in \mathbb{Z}_n$ there exists exactly one $c_{jk} := i \in \mathbb{Z}_q$ with $j \in P_{ik}$. Then $C := \{(c_{j0}, \ldots, c_{j,n-1}) \in \mathbb{Z}_q^n \mid j \in \mathbb{Z}_M\}$ is a code of cardinality $|C| \leq M$ which by our assumption is not $s$-surjective with radius $s-2$. Hence, there is an $s$-tupel $(k_1, \ldots, k_s) \in \mathbb{Z}_n^s$ of pairwise distinct coordinates and an $s$-tupel $(x_1, \ldots, x_s) \in \mathbb{Z}_q^s$ such that for every $j \in \mathbb{Z}_M$ the equation $c_{jk_i} = x_i$ holds for at most one $i \in \{1, \ldots, s\}$. Consequently, $(P_{x_i k_i})_{i \in \{1,\ldots,s\}}$ is the desired $s$-transversal. □

The next theorem contains a powerful new recursive bound on $K_q(n, R)$, confer the table in Section 5.

**Theorem 6.** *If $r < s$ then $\sigma_{q+1}(n+1, s+1; r+1) \geq \min\{2(q+1), \sigma_q(n, s; r) + 1\}$. Especially $K_{q+1}(n+1, R+1) \geq \min\{2(q+1), K_q(n, R) + 1\}$ if $R < n$.*

*Proof.* In case of $s - r = 1$, the theorem follows from $\sigma_q(n, r+1; r) = q$. Assume $s - r \geq 2$. Let $C \subset \mathbb{Z}_{q+1}^{n+1}$ be a code of cardinality $\min\{2q+1, \sigma_q(n, s; r)\}$. For $i \in \mathbb{Z}_{n+1}$, $z \in \mathbb{Z}_{q+1}$ we set $C_{iz} = \{(y_0, \ldots, y_n) \in C \mid y_i = z\}$ and

$$f : \mathbb{Z}_{q+1} \to \mathbb{Z}_q, z \mapsto \begin{cases} z & \text{if } z < q \\ 0 & \text{if } z = q. \end{cases}$$

There is a $z \in \mathbb{Z}_{q+1}$ such that $|C_{nz}| \leq 1$. W.l.o.g. let $z = q$ and $C_{nq} \subset \bigcap_{i=0}^{n-1} C_{iq} \neq \emptyset$. Put

$$C' := \left\{ (f(y_0), \ldots, f(y_{n-1})) \in \mathbb{Z}_q^n \mid (y_0, \ldots, y_n) \in C \setminus \bigcap_{i=0}^{n-1} C_{iq} \right\}.$$

Since $|C'| < |C| \leq \sigma_q(n, s; r)$, the code $C'$ is not $s$-surjective with radius $r$. Hence, there is an $s$-tupel $k \in \mathbb{Z}_n^s$ of pairwise distinct coordinates and an $s$-tupel $x \in \mathbb{Z}_q^s$ such that for every $c \in C'$ the equation $c_{k_i} = x_i$ holds for less than $s - r$ coordinates. Put $\bar{k} := (k, n) \in \mathbb{Z}_{n+1}^{s+1}$ and $\bar{x} := (x, q) \in \mathbb{Z}_{q+1}^{s+1}$. Then for every $\bar{c} \in C$ the equation $\bar{c}_{\bar{k}_i} = \bar{x}_i$ holds for less than $s - r$ coordinates. Thus, $C$ is not $(s+1)$-surjective with radius $r+1$. □

# 3 On Rodemich's Bound

As a first application of Theorem 5 we give a new proof of Rodemich's bound (1) in the following slightly improved version.

**Theorem 7.** *Let $p \in \mathbb{Z}_{n-1}$ such that $q \equiv p \pmod{n-1}$. Then*

$$K_q(n, n-2) \geq \frac{q^2 - p^2}{n-1} + p. \tag{3}$$

*Proof.* Consider an $(n, M, q)$-partition matrix $\mathcal{P}$ without $n$-transversal. By Theorem 5 (with $s = n$) it suffices to show, that $M$ can be lower-bounded by the right-hand side of (3).

We define the notion of a minimal $s$-transversal in $\mathcal{P}$ recursively as follows: a 0-transversal is minimal and an $s$-transversal $\mathcal{T}_s = (P_{x_i k_i})_{i \in \{1,\ldots,s\}}$ with $s \geq 1$ is minimal, if it contains a minimal $(s-1)$-transversal, and if among all $s$-transversals with this property,

$$l(\mathcal{T}_s) := \left| \bigcup_{i=1}^{s} P_{x_i k_i} \right| = \sum_{i=1}^{s} |P_{x_i k_i}|$$

is minimal. Let $t$ be the largest integer, such that there exists a minimal $t$-transversal $\mathcal{T}_t = (P_{x_i k_i})_{i \in \{1,\ldots,t\}}$ in $\mathcal{P}$. We have $1 \leq t \leq n-1$ since $\mathcal{P}$ is supposed to be without $n$-transversal. For every $s \in \{1, \ldots, t\}$ we set $A_s = P_{x_s k_s}$ and may assume that $\mathcal{T}_t = (A_1, \ldots, A_t)$ is ordered in such a way, that $\mathcal{T}_s := (A_1, \ldots, A_s)$ is a minimal $s$-transversal. Moreover, for every $s \in \{1, \ldots, t\}$ set $l_s = |A_s|$ and $L_s = l(\mathcal{T}_s) = l_1 + \ldots + l_s$ as well as $L_0 = 0$. By $t \leq n-1$ there is a column $k \in \mathbb{Z}_n$ of $\mathcal{P}$, which is not used in $\mathcal{T}_t$. Without loss

$$|P_{0k}| \leq |P_{1k}| \leq \ldots \leq |P_{q-1,k}|. \tag{4}$$

Now let $u$ be the largest integer $\leq t$ with $L_u < q$. We have $u \leq t-1$ $(\leq n-2)$ since otherwise $u = t$ and at least one set of column $k$ is disjoint to $A_1 \cup \ldots \cup A_t$, which means, that $\mathcal{T}_t$ could be extended to a $(t+1)$-transversal, contradicting the maximality of $t$. We now claim that at least $q - L_u$ sets of column $k$ have cardinality $\geq q - L_u$: if this was not true, there would be $\geq L_u + 1$ sets in column $k$ with cardinality $< q - L_u$, that is, we could extend the transversal $\mathcal{T}_u$ by some set of column $k$ to a transversal $\mathcal{T}'$ with $l(\mathcal{T}') < q \leq L_{u+1} = l(\mathcal{T}_{u+1})$, contradicting the minimality of $\mathcal{T}_{u+1}$. Similarly, for each $s \in \mathbb{Z}_u$ there are at least $q - L_s$ sets of column $k$ with cardinality $\geq l_{s+1}$, for otherwise $\mathcal{T}_{s+1}$ would not be minimal. By (4) we have $|P_{ik}| \geq q - L_u$ if $i \geq L_u$ and $|P_{ik}| \geq l_{s+1}$ if $i \geq L_s, s \in \mathbb{Z}_u$. Thus we obtain

$$M = \left| \bigcup_{i=0}^{q-1} P_{ik} \right| = \sum_{s=0}^{u-1} \sum_{L_s \leq i < L_{s+1}} |P_{ik}| + \sum_{L_u \leq i < q} |P_{ik}| \geq \sum_{s=0}^{u-1} l_{s+1}^2 + (q - L_u)^2.$$

The right-hand side of this inequality is a sum of the squares of $u+1$ integers which theirselves sum up to $q$. It is well-known that such a sum is minimal if $u+1$ is maximal (i.e. $u+1 = n-1$) and the mutual distances of the integers are minimal, that is,

$$M \geq (n-1-p) \left\lfloor \frac{q}{n-1} \right\rfloor^2 + p \left\lceil \frac{q}{n-1} \right\rceil^2 = \frac{q^2}{n-1} + p - \frac{p^2}{n-1}.$$

□

The following new result improves Rodemich's bound (1) iff $5 \leq n < q \leq 2n-4$.

**Theorem 8.** $K_q(n, n-2) \geq 3q - 2n + 2$.

*Proof.* Let $\mathcal{P} = (P_{ik})$ be an $(n, 3q-2n+1, q)$-partition matrix. By Theorem 5 it suffices to show, that $\mathcal{P}$ has an $n$-transversal. Choose a transversal $\mathcal{T}$ of maximal length (say $t$) consisting of sets with cardinality $\leq 1$. If $t = n$ then the claim follows, so let $t < n$. W.l.o.g. the subsets of $\mathcal{T}$ are from the first $t$ columns of $\mathcal{P}$. Let $p$ be the number of 1-sets in $\mathcal{T}$. Consider column $k \in \mathbb{Z}_n \setminus \mathbb{Z}_t$ and let $a$ be the number of 1-sets in this column. The maximality of $\mathcal{T}$ implies that there is no empty set in this column and $a \leq p$. Hence, the number of sets of cardinality $\geq 3$ in this column is $\leq (3q-2n+1-a) - 2(q-a) \leq q - 2n + p + 1$. Recursively, we now define a sequence $(\mathcal{T}_s)_{s \in \{t,\ldots,n\}}$ of $s$-transversals consisting of sets of cardinality $\leq 2$ only. Set $\mathcal{T}_t := \mathcal{T}$. Assume $\mathcal{T}_{s_0}$ is already defined for an $s_0$ with $t \leq s_0 < n$.

Then the number of sets of a column not used in $\mathcal{T}_{s_0}$, which are not disjoint to all sets of $\mathcal{T}_{s_0}$ is $\leq p + 2(s_0 - t)$. Hence, the number of 2-sets in this column, which are disjoint to all sets of $\mathcal{T}_{s_0}$ is

$$\geq q - (q - 2n + p + 1) - (p + 2(s_0 - t)) \geq 2(n - s_0) - 1 \geq 1.$$

Choose such a set and add it to $\mathcal{T}_{s_0}$ in order to obtain $\mathcal{T}_{s_0+1}$, still consisting of sets of cardinality $\leq 2$. Finally, $\mathcal{T}_n$ is the desired $n$-transversal. □

Bound (1) implies for instance $K_{3n}(2n+1, 2n-1) \geq 9n/2$, while Theorem 8 improves it to $K_{3n}(2n+1, 2n-1) \geq 5n$.

# 4 Exact Values

The application of partition matrices and their transversals is also suitable for obtaining specific lower bounds. For instance the following result, together with Theorem 6, leads to six new exact values on $K_q(q, q-2)$, extending equality in (2) to $q \leq 10$.

**Theorem 9.** *$K_5(5, 3) = 9$.*

*Proof.* The upper bound follows from (2). For the lower bound we will prove that every $(5, 8, 5)$-partition matrix $\mathcal{P}$ has a 5-transversal by considering several cases. We may assume that $\mathcal{P}$ does not contain an empty set since otherwise the bound $K_5(4, 2) \geq 9$ (see (1)) leads to a 5-transversal. Let $t$ be the maximal length of a transversal $\mathcal{T}$ consisting of sets of cardinality 1. Clearly, $t \geq 2$ since each column of $\mathcal{P}$ contains at least two sets of cardinality 1. If $t \geq 4$ then the claim follows easily. So it remains to consider the cases $t = 2$ and $t = 3$.

Let $t = 2$. Because of the maximality of $\mathcal{T}$ every column of $\mathcal{P}$ consists of three 2-sets and the same two 1-sets, say $\{6\}$ and $\{7\}$. W.l.o.g. $\{6\} = P_{3k}$ and $\{7\} = P_{4k}$ for $k \in \mathbb{Z}_5$. Delete row 3 and 4 to obtain a $(5, 6, 3)$-partition matrix. By Theorem 2 and 5, it has a 3-transversal. Thus, $\mathcal{P}$ has a 5-transversal.

Let $t = 3$. For $k \in \mathbb{Z}_5$ and $x \in \mathbb{Z}_8$ set

$$s_k(x) = \begin{cases} 1 & \text{if } \{x\} \text{ occurs in column } k \\ 0 & \text{otherwise} \end{cases}$$

and $s(x) = \sum_{k=0}^{4} s_k(x)$. W.l.o.g. let $k_5, k_6, k_7 \in \mathbb{Z}_5$ be pairwise distinct columns such that $s_{k_5}(5) = s_{k_6}(6) = s_{k_7}(7) = 1$ and

$$s(7) \geq s(6) \geq s(5). \tag{5}$$

Set $\{k', k''\} = \mathbb{Z}_5 \setminus \{k_5, k_6, k_7\}$ and $s_k = s_k(5) + s_k(6) + s_k(7)$ for all $k \in \mathbb{Z}_5$. Clearly, $s_{k_5}, s_{k_6}, s_{k_7} \geq 1$. The maximality of $\mathcal{T}$ implies $s_{k'}, s_{k''} \geq 2$.

Next, we proof the following auxiliary statement: $s(7) \geq 4$, $s(6) \geq 3$, if there is a column $l \in \mathbb{Z}_5$ such that $s_l = 1$ then $s(6) \geq 4$. First assume there is an $l$ with $s_l = 1$. Clearly, $l \in \{k_5, k_6, k_7\}$. Let $l = k_a$, i.e. $s_l(a) = 1$, and $\{a', a''\} = \{5, 6, 7\} \setminus \{a\}$. Since every column of $\mathcal{P}$ contains at least two sets of cardinality 1, there is an $x \in \mathbb{Z}_5$ such that $s_l(x) = 1$. Set $\{l', l''\} = \{k_5, k_6, k_7\} \setminus \{l\}$. The maximality of $\mathcal{T}$ implies $s_{k'}(a) = s_{k''}(a) = 0$

and, hence, $s_{k'}(a') = s_{k''}(a') = s_{k'}(a'') = s_{k''}(a'') = 1$. The maximality also implies $s_{l'}(a') = s_{l''}(a') = s_{l'}(a'') = s_{l''}(a'') = 1$ as well as $s_{l'}(a) = s_{l''}(a) = 0$. Consequently, $s(a) = 1$, $s(a') = s(a'') = 4$. Finally, $a = 5$ and $\{a', a''\} = \{6, 7\}$ follow by (5), implying $s(7), s(6) \geq 4$. Now assume there is no $l$ with $s_l = 1$ then $s(5) + s(6) + s(7) = \sum_{k=0}^{4} s_k \geq 5 \cdot 2 = 10$. Recall $s(7) \leq 5$. By (5) it turns out that $s(7) \geq 4$ and $s(6) \geq 3$, finishing the proof of the auxiliary statement.

W.l.o.g. let $6, 7 \in P_{3k} \cup P_{4k}$ for all $k \in \mathbb{Z}_5$. Delete row 3 and 4 of $\mathcal{P}$ and add, if necessary, some elements of $\mathbb{Z}_6$ to obtain a $(5, 6, 3)$-partition matrix $\mathcal{P}'$. By Theorem 2 and 5, it has a 3-transversal, say $P'_{02}, P'_{03}, P'_{04}$. W.l.o.g. let $5 \notin P'_{03} \cup P'_{04}$ and $s_0(7) = 1$ by the auxiliary statement. If $s_1(6) = 1$ or $s_0(6) = s_1(7) = 1$ then $\{7\}, \{6\}, P_{02}, P_{03}, P_{04}$ is the desired 5-transversal. If $s_1(6) = s_0(6) = 0$ then the auxiliary statement implies $s_2(6) = 1$, since $s(6) \geq 3$, and $s_1(5) = 1$, since $s_1 \geq 2$, so that $\{7\}, \{5\}, \{6\}, P_{03}, P_{04}$ is the desired 5-transversal. If $s_1(6) = s_1(7) = 0$ then $s_1 = s_1(5) = 1$ and $s_2(6) = 1$ by the auxiliary statement, so that again $\{7\}, \{5\}, \{6\}, P_{03}, P_{04}$ is the desired 5-transversal. □

**Corollary 10.** *$K_q(q, q-2) = q - 2 + \sigma_2(q, 2; 0)$ if $q \leq 10$.*

*Proof.* Apply (2), Theorem 6 and Theorem 9. □

It appears to the authors, that the method can improve many of the currently best known lower bounds on $K_q(n, n-2)$ in Kéri's tables [5]. The same holds for lower bounds on $\sigma_q(n, s; s-2)$. In particular, we announce $K_5(4, 2) = 11$ and a non-computational proof of Theorem 2.

# 5 A Table with New Lower Bounds

Table 1 collects all new lower bounds on $K_q(n, R)$ from this paper. Entries in bold are exact. We use the inequality

$$K_q(n_1 + n_2, R_1 + R_2 + 1) \geq \min\{K_q(n_1, R_1), K_q(n_2, R_2)\} \tag{6}$$

due to Bhandari, Durairajan [2].

*Table 1.* New lower bounds on $K_q(n,R)$

| $K_q(n,R)$ | Reference | Old Lower Bound [5] | New Lower Bound | Upper Bound [5] |
|---|---|---|---|---|
| $K_5(5,3)$ | Theorem 9 | 8 | **9** | 9 |
| $K_5(9,6)$ | Inequality (6) | 8 | 9 | 15 |
| $K_5(10,7)$ | Inequality (6) | 8 | 9 | 10 |
| $K_6(6,4)$ | Theorem 6 | 8 | **10** | 10 |
| $K_7(5,3)$ | Theorem 6 | 13 | 14 | 17 |
| $K_7(6,4)$ | Theorem 6 | 10 | 11 | 15 |
| $K_7(7,5)$ | Theorem 6 | 9 | **11** | 11 |
| $K_7(9,6)$ | Theorem 6 | 13 | 14 | 37 |
| $K_7(10,7)$ | Inequality (6) | 13 | 14 | 37 |
| $K_8(6,4)$ | Theorem 6 | 13 | 15 | 20 |
| $K_8(7,5)$ | Theorem 6 | 11 | 12 | 16 |
| $K_8(8,6)$ | Theorem 6 | 10 | **12** | 12 |
| $K_9(6,4)$ | Theorem 6 | 17 | 18 | 24 |
| $K_9(7,5)$ | Theorem 6 | 14 | 16 | 21 |
| $K_9(8,6)$ | Theorem 6 | 12 | 13 | 17 |
| $K_9(9,7)$ | Theorem 6 | 11 | **13** | 13 |
| $K_{10}(7,5)$ | Theorem 6 | 17 | 19 | 26 |
| $K_{10}(8,6)$ | Theorem 6 | 15 | 17 | 22 |
| $K_{10}(9,7)$ | Theorem 6 | 13 | 14 | 18 |
| $K_{10}(10,8)$ | Theorem 6 | 12 | **14** | 14 |
| $K_{11}(7,5)$ | Theorem 6 | 21 | 22 | 31 |
| $K_{11}(8,6)$ | Theorem 6 | 18 | 20 | 27 |
| $K_{12}(6,4)$ | Theorem 7 | 29 | 30 | 41 |
| $K_{12}(8,6)$ | Theorem 6 | 21 | 23 | 32 |
| $K_{13}(6,4)$ | Theorem 7 | 34 | 35 | 46 |
| $K_{14}(7,5)$ | Theorem 7 | 33 | 34 | 48 |
| $K_{15}(7,5)$ | Theorem 7 | 38 | 39 | 54 |
| $K_{16}(7,5)$ | Theorem 7 | 43 | 44 | 60 |
| $K_{16}(8,6)$ | Theorem 7 | 37 | 38 | 56 |
| $K_{17}(6,4)$ | Theorem 7 | 58 | 59 | 73 |
| $K_{17}(8,6)$ | Theorem 7 | 42 | 43 | 63 |
| $K_{18}(6,4)$ | Theorem 7 | 65 | 66 | 80 |
| $K_{18}(8,6)$ | Theorem 7 | 47 | 48 | 70 |
| $K_{19}(8,6)$ | Theorem 7 | 52 | 53 | 77 |
| $K_{20}(7,5)$ | Theorem 7 | 67 | 68 | 89 |
| $K_{21}(7,5)$ | Theorem 7 | 74 | 75 | 98 |

# References


[1] W. Benz, On structures $\mathrm{T}(t, q, r, n)$, *Annals of Discrete Math.* 52 (1992) 25-36.

[2] M.C. Bhandari, C. Durairajan, A note on bounds for $q$-ary covering codes, *IEEE Trans. Inform. Theory* 42 (1996) 1640-1642.

[3] A. Brace, D.E. Daykin, Sperner type theorems for finite sets, *Proc. British Combinatorial Conf.* (1972) 18-37.

[4] G. Cohen, I. Honkala, S. Litsyn, A. Lobstein, *Covering Codes*, North Holland, Amsterdam, 1997.

[5] G. Kéri, *Tables for Covering Codes*, http://www.sztaki.hu/~keri/codes/, accessed 16 December 2007.

[6] G. Kéri, P.R.J. Östergård, On the covering radius of small codes, *Studia Sci. Math. Hungar.* 40 (2003) 243-256.

[7] G. Kéri, P.R.J. Östergård, Bounds for covering codes over large alphabets, *Des. Codes Cryptogr.* 137 (2005) 45-60.

[8] G. Kéri, P.R.J. Östergård, Further results on the covering radius of small codes, *Discrete Math.* 307 (2007) 69-77.

[9] D.J. Kleitman, J.H. Spencer, Families of $k$-independent sets, *Discrete Math.* 6 (1973) 255-262.

[10] P.R.J. Östergård, Upper bounds for $q$-ary covering codes, *IEEE Trans. Inform. Theory* 37 (1991) 660-664; 37 (1991) 1738.

[11] P.R.J. Östergård, Construction methods for mixed covering codes, in: M. Gyllenberg, L.E. Persson (Eds.), *Analysis, Algebra, and Computers in Mathematical Research*, Dekker, New York, 1994, pp. 387-408.

[12] E.R. Rodemich, Coverings by rook domains, *J. Combin. Theory Ser. A* 9 (1970) 117-128.