ON FABRY'S GAP THEOREM

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ABSTRACT. By combining Turán's proof of Fabry's gap theorem with a gap theorem of P. Szüsz we obtain a gap theorem which is more general then both these theorems.

Hadamard's classical gap theorem states that if $f(z) = \sum_{l=1}^{\infty} a_l z^{k_l}$ is a power series with radius of convergence 1, and $\frac{k_{l+1}}{k_l} \geq \theta > 1$, the circle $|z| = 1$ is the natural boundary of f. The condition on the growth of the k_l was lessened by Fabry to $k_l/l \to \infty$ without changing the conclusion(see [1]). Using his powersum method, P. Turán gave a simple proof of this theorem(see [2] or [3], section 20). By a completely elementary argument, P. Szüsz[4] proved the following theorem:

Theorem 1. *Suppose that $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is a power series with radius of convergence 1. Assume that there is a subsequence k_l of indices, such that* $\limsup \sqrt[k_l]{a_{k_l}} = 1$*, and a constant $\theta > 0$, such that $a_k = 0$ for all $k \neq k_l$, $|k - k_l| < \theta k_l$. Then the circle $|z| = 1$ is the natural boundary of f.*

Note that this theorem is stronger then Hadamard's gap theorem, but neither implies nor is implied by Fabry's gap theorem. In this note we will combine P. Turán's proof of Fabry's gap theorem with the idea of P. Szüsz to obtain the following theorem, which implies both gap theorems.

Theorem 2. *Suppose that $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is a power series with radius of convergence 1. Assume that there is a subsequence k_l of indices, such that* $\limsup \sqrt[k_l]{a_{k_l}} = 1$*, and a constant $\theta > 0$, such that the following holds true: If N_l denotes the number of indices k within the interval $[(1-\theta)k_l, (1+\theta)k_l]$, such that $a_k \neq 0$, then $N_l/k_l \to 0$. Then the circle $|z| = 1$ is the natural boundary of f.*

To obtain Fabry's gap theorem from this theorem, we can choose the sequence of all nonvanishing coefficents, to deduce Szüsz' gap theorem, note that under the assumptions of theorem 1 we have $N_l = 1$ for any l.

To prove theorem 2, it suffices to show that f is singular in 1. Assume that f was regular in 1. Then there was some $c > 0$, such that f is regular in $\{|z - 1| < c\} \cup \{|z| < 1\}$. Choose $\delta < \theta/2$, and $\varphi > 0$ so small, that f is regular within the domain $|z| < 1 + c/2, |\arg z| < 2\varphi$. Then we will consider f on the arc $\mathcal{C} = \{|z| = \delta, |\arg z| < \varphi\}$. First we give an upper bound for $|f^{(k)}(z)|$ on $\mathcal{C}$. We may assume that δ is so small, that around any point of $\mathcal{C}$, there is a disc with radius $1 + \delta$, such that f is holomorphic within this disc. By the standard integral estimate we obtain $|f^{(k)}(z)| < k!(1 + \delta/2)^{-k} < k!/2$ for k sufficently large.

To obtain a lower bound, we want to use Turán's powersum technic. Let k be an index occuring in the sequence of our theorem and choose m to be the greatest integer such that $\left[\frac{m}{1-\delta}\right] \leq k$. Then we have

$$\frac{f^{(m)}(z)}{m!} = \sum_{\nu=0}^{\infty} \binom{\nu}{m} z^{\nu-m} a_\nu = \sum_{\nu=m}^{(1+\delta)k} \binom{\nu}{m} z^{\nu-m} a_\nu + \sum_{\nu>(1+\delta)k} \binom{\nu}{m} z^{\nu-m} a_\nu$$

Now we restrict z to be an element of $\mathcal{C}$. We bound the second sum first. We have by the restriction $\nu > (1+\delta)k$

$$\frac{\binom{\nu+1}{m}}{\binom{\nu}{m}} \delta < 2/3$$

Together with the bound $a_\nu < (6/5)^\nu$, valid for ν sufficiently large, we see that the second sum is $< 1/2$.

Combining these estimates we obtain the following inequality:

$$\max_{z\in\mathcal{C}} \left| \sum_{\nu=m}^{(1+\delta)k} \binom{\nu}{m} z^{\nu-m} a_\nu \right| < 1 \tag{1}$$

The left hand side can be bounded from below using the continuos version of the first main theorem in the powersum theory (see [3], section 6):

Theorem 3. *For $1 \leq \nu \leq n$ let α_ν be complex numbers, λ_ν real numbers. Then we have for every $\mu > 0$ the inequality*

$$\max_{0\leq x\leq \mu} \left| \sum_{\nu=1}^{n} \alpha_\nu e^{i\lambda_\nu x} \right| \geq \left(\frac{\mu}{4e\pi}\right)^n \max_{0\leq x\leq 2\pi} \left| \sum_{\nu=1}^{n} \alpha_\nu e^{i\lambda_\nu x} \right|$$

Applying this to the left hand side of (1), we obtain the bound

$$\max_{|z|=\delta} \left| \sum_{\nu=m}^{(1+\delta)k} \binom{\nu}{m} z^{\nu-m} a_\nu \right| < \left(\frac{2e\pi}{\varphi}\right)^{N_l}$$

The maximum of this sum is at least its quadratic mean, and the latter can be computated using Parseval's equality, thus we get the bound

$$\sum_{\nu=m}^{(1+\delta)k} \binom{\nu}{m}^2 \delta^{2(\nu-m)} |a_\nu|^2 < \left(\frac{2e\pi}{\varphi}\right)^{2N_l}$$

Neglecting all terms on the left hand side except $\nu = k$, we get

$$\binom{k}{m}^2 \delta^{2(k-m)} |a_k|^2 < \left(\frac{2e\pi}{\varphi}\right)^{2N_l}$$

By assumption we have $N_l = o(k_l)$, thus the right hand side is $e^{o(m)}$. The same is true for $|a_k|$, and computing the binomial coefficient using Stirling's formula, we get the final contradiction

$$\frac{1}{(1-\delta)^{2m}} = e^{o(m)}$$

which finishes our proof.

References

[1] L. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, Berlin 1929

[2] P. Turán, *On the gap-theorem of Fabry*, Acta Math. Sci. Acad. Hung. 1, 21-29 (1947)

[3] P. Turán, *On a new method of analysis and its applications*, Wiley-Interscience Publication. New York etc. (1984).

[4] P. Szüsz, *On Hadamard's gap theorem*, Studies in pure mathematics to the memory of Paul Turán, Budapest 1983, 733-736

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