THE NUMBER OF k-DIGIT FIBONACCI NUMBERS

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Define $a(k)$ to be the number of k-digit Fibonacci numbers. For $n > 5$, we have $1.6F_{n-1} < F_n < 1.7F_{n-1}$. Thus if F_n is the least k-digit Fibonacci number, we have $F_{n+5} > 1.6^5 F_n > 10.48 \cdot 10^{k-1}$. On the other hand $F_{n+3} < 1.7^4 F_{n-1} < 1.7^4 \cdot 10^{k-1} < 8.36 \cdot 10^{k-1}$. Hence F_{n+5} always has at least $k+1$ digits, but F_{k+3} always has k digits. Thus for $k > 1$ we always have $a(k) = 4$ or $a(k) = 5$. Define $A(x)$ to be the number of $k \leq x$, such that $a(k) = 5$. Then Guthmann[1] proved the following theorem.

Theorem 1. *For $x \to \infty$ we have*

$$A(x) = \alpha x + O(1)$$

where

$$\alpha = \log 10 / \log((1 + \sqrt{5})/2) - 4 = 0.78497\ldots$$

His proof uses Baker's bound on linear forms in logarithms. Here we will give a very short proof of this statement and generalize it to residue classes. Since except for $k = 1$ we have $a(k) = 4$ or 5, we get

$$\#\{n | F_n < 10^x\} = \sum_{k \leq x} a(k) = 4(x - A(x)) + 5A(x) + O(1)$$

On the other hand we have $F_n \sim \frac{1}{\sqrt{5}}\varphi^n$, thus the left hand side is $x\frac{\log 10}{\log \varphi} + O(1)$. Now solving for $A(x)$ gives the theorem.

Now define $A(x, q, l)$ to be the number of $k \leq x, k \equiv l \pmod{q}$, such that $a(k) = 5$. With this notation we claim the following theorem.

Theorem 2. *For any fixed q we have*

$$A(x, q, l) \sim \frac{\alpha}{q} x$$

where α is defined as above.

We first note that $F_{n+4}/F_n \to \varphi^4$. If F_n is the least Fibonacci number with k digits, then $a(k) = 5$ if and only if $F_{n+4} < 10^k$. Now let $\epsilon > 0$ be fixed. Then we consider 3 cases:

(1) $10^{k-1} < F_n < \left(\frac{10}{\varphi^4} - \epsilon\right) 10^{k-1}$
If n is sufficiently large, this implies $F_{n+4} < 10^k$, thus $a(k) = 5$.

(2) $\left(\frac{10}{\varphi^4} - \epsilon\right) 10^{k-1} < F_n < \left(\frac{10}{\varphi^4} + \epsilon\right) 10^{k-1}$
In this case we might have $a(k) = 4$ or $a(k) = 5$.

(3) $F_n > \left(\frac{10}{\varphi^4} + \epsilon\right) 10^{k-1}$
In this case we have for n sufficiently large $F_{n+4} > 10^k$, thus only $F_n, \ldots F_{n+3}$ have k digits which implies $a(k) = 4$. We also note that in this case we have $F_n < (\varphi + \epsilon)10^{k-1}$, since otherwise F_{n-1} would also have k digits.

If we only consider case 1, we get a lower bound for $A(x, q, l)$, thus we have for $x > x_0(\epsilon)$ the estimate

$$A(x,q,l) \geq \#\{k \leq x, k \equiv l \pmod q | \exists n : 10^{k-1} < F_n < \left(\frac{10}{\varphi^4} - \epsilon\right) 10^{k-1}\}$$

We set $k = k'q + l$, and taking logarithms we get

$$\begin{aligned} A(x,q,l) \ &\geq \ \#\big\{k' \leq \frac{x-l}{q} | \exists n : \\ &\qquad (k'q + l - 1)\log 10 + \epsilon < n \log\varphi < (k'q + l)\log 10 - 4\log\varphi - \epsilon\big\} \end{aligned}$$

which is equivalent to

$$A(x,q,l) \geq \#\{k' \leq \frac{x-l}{q} | \exists n : n\log\varphi - l\log 10 + 4\log\varphi + \epsilon < k'q\log 10 < n\log\varphi - l\log 10 + \log 10 - \epsilon\}$$

Since $\frac{q\log 10}{\log\varphi}$ is irrational, the fractional part of $\frac{k'q\log 10}{\log\varphi}$ is uniformly distributed (mod 1) if k' runs over all integers. k' is counted if and only if the fractional part of $\frac{k'q\log 10}{\log\varphi}$ is contained is some interval of length $\frac{\log 10 - 4\log\varphi - 2\epsilon}{\log\varphi} \geq \alpha - 5\epsilon$, hence for $y > y_0$ the the number of $k' < y$ with $a(k'q+l) = 5$ is $\geq (\alpha - 6\epsilon)y$. If $k' < \frac{x-l}{q}$, then $k \leq x$, thus we obtain the lower bound $A(x,q,l) \geq (\alpha - 6\epsilon)\frac{x}{q}$. In the same way we get the upper bound $A(x,q,l) \leq (\alpha + 6\epsilon)\frac{x}{q}$, if $\epsilon \to 0$, we obtain the statement of theorem 2.

I would like to thank the referee for correcting the proof of theorem 2.

References

[1] A. Guthmann, *Wieviele k-stellige Fibonaccizahlen gibt es?* Arch. Math. 59, No.4, 334-340 (1992)

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