Homomorphisms from a finite groups into wreath products

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Abstract. Let G be a finite group, A a finite abelian group. Each homomorphism $\varphi : G \to A \wr S_n$ induces a homomorphism $\overline{\varphi} : G \to A$ in a natural way. We show that as φ is chosen randomly, then the distribution of $\overline{\varphi}$ is close to uniform. As application we prove a conjecture of T. Müller on the number of homomorphisms from a finite group into Weyl groups of type D_n.

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Let G be a finite group, A a finite abelian group. In this article we consider the number of homomorphisms $G \to A \wr S_n$, where n tends to infinity. These numbers are of interest, since they encode information on the isomorphism types of subgroups of index n, confer [2], [3]. If $\varphi : G \to A \wr S_n$ is a homomorphism, we can construct a homomorphism $\overline{\varphi} : G \to A$ as follows. We represent the element $\varphi(g) \in A \wr S_n$ as $(\pi; a_1, \dots, a_n)$, where $\pi \in A_n$ and $a_i \in A$, and then define $\overline{\varphi}(g) = \prod_{i=1}^n a_i$. The fact that $\overline{\varphi}$ is a homomorphism follows from the fact that A is abelian and the definition of the product within a wreath product. In this article we prove the following.

Theorem 1. *Let G be a finite group of order d, A a finite abelian group. Define the distribution function δ_n on $\mathrm{Hom}(G, A)$ as the image of the uniform distribution on $\mathrm{Hom}(G, A \wr S_n)$ under the map $\varphi \mapsto \overline{\varphi}$. Then there exist positive constants c, C, independent of n, such that $\|\delta_n - u\|_\infty < Ce^{-cn^{1/d}}$, where u is the uniform distribution, and $\|\cdot\|_\infty$ denotes the supremum norm.*

As an application we prove the following, which confirms a conjecture by T. Müller.

Corollary 2. *For a finite group G there exists a constant $c > 0$, such that if W_n denotes the Weyl group of type D_n, then*

$$|\,\mathrm{Hom}(G, W_n)| = \left(\frac{1}{1 + s_2(G)} + \mathcal{O}(e^{-cn^{1/d}})\right) |\,\mathrm{Hom}(G, C_2 \wr S_n)|$$

This assertion was proven by T. Müller under the assumption that G is cyclic (confer [1, Proposition 3]). Different from his approach we do not enumerate homomorphisms φ with given image $\overline{\varphi}$, but directly work with the distribution of $\overline{\varphi}$, that is, we obtain the relation between $|\operatorname{Hom}(G, W_n)|$ and $|\operatorname{Hom}(G, C_2 \wr S_n)|$ without actually computing these functions.

Denote by $\pi : A \wr S_n \to S_n$ the canonical projection onto the active group. The idea of the proof is to stratify the set $|\operatorname{Hom}(G, A \wr S_n)|$ according to $\pi \circ \varphi \in \operatorname{Hom}(G, S_n)$. It turns out that in strata such that $\pi \circ \varphi(G)$ viewed as a permutation group on $\{1, \dots, n\}$ has a fixed point the distribution of $\overline{\varphi}$ is actually uniform, while the probability of having no fixed point is very small.

Lemma 3. *Let $\varphi_1 : G \to S_n$ be a homomorphism. Let $\varphi : G \to A \wr S_n$ be a homomorphism chosen at random with respect to the constraint $\pi \circ \varphi = \varphi_1$. If the action of G on $\{1, \dots, n\}$ induced by φ_1 has a fixed point, then $\overline{\varphi}$ is uniformly distributed on* $\operatorname{Hom}(G, A)$.

Proof. Without loss we may assume that the point 1 is fixed. A homomorphism $\varphi : G \to A \wr S_n$ with $\pi \circ \varphi = \varphi_1$ can be viewed as a pair of homomorphisms $\psi_1 : G \to A \wr S_1$, $\psi_2 : G \to A \wr S_{n-1}$, where the projection of ψ_2 to S_{n-1} is given by the restriction of φ_1 to $\{2, \dots, n\}$. If we chose φ at random, then ψ_1 and ψ_2 are independent, and ψ_1 is clearly uniformly distributed. Now $\overline{\varphi}$ is the sum of two independent random elements in $\operatorname{Hom}(G, A)$, one of which is uniformly distributed. Since $\operatorname{Hom}(G, A)$ with pointwise addition is an abelian group, we conclude that $\overline{\varphi}$ is uniformly distributed. □

To bound the number of homomorphisms φ for which $\pi \circ \varphi$ has no fixed point we need the following, which is contained in [2, Proposition 1], in particular the equality of equations (8) and (9) in that article.

Lemma 4. *Let G be a group, A a finite abelian group, U a subgroup of index k, $\varphi_1 : G \to S_k$ the permutation representation given by the action of G on G/U. Then the number of homomorphisms $\varphi : G \to A \wr S_k$ with $\pi \circ \varphi = \varphi_1$ equals* $|A|^{k-1} |\operatorname{Hom}(U, A)|$.

We use this to prove the following.

Lemma 5. *Let G be a group of order d, A a finite abelian group, $\varphi : G \to A \wr S_n$ be a homomorphism chosen at random. Then there is a constant $c > 0$, depending only on G, such that the probability that $\pi \circ \varphi(G)$ has no fixed points is $\mathcal{O}(e^{-cn^{1/d}})$.*

Proof. Let $U_1, \dots, U_\ell$ be a complete list of subgroups of G, where $U_\ell = G$. To determine a homomorphism $\varphi : G \to A \wr S_n$ we first have to chose a homomorphism $\varphi_1 : G \to S_n$, and then count the number of ways in which this homomorphism can be extended to a homomorphism to $A \wr S_n$. Suppose that the action of G on $\{1, \dots, n\}$ induced by φ_1 has m_i orbits on which G acts similar to the action of G on G/U. Then by the previous lemma we find that there are

$$\prod_{i=1}^{\ell}\left(|A|^{(G:U_i)-1}|\operatorname{Hom}(U_i,A)|\right)^{m_i}$$

possibilities to extend φ_1. The number of ways to choose φ_1 given $m_1,\dots,m_\ell$ is $\frac{n!}{\prod_{i=1}^{\ell} m_i!(G:U_\ell)^{m_i}}$, hence, we obtain

$$|\operatorname{Hom}(G,A\wr S_n)| = n!\sum_{\substack{m_1,\dots,m_\ell\\ m_1+\dots+m_\ell=n}}\prod_{i=1}^{\ell}\frac{1}{m_i!}\left(\frac{|A|^{(G:U_i)-1}|\operatorname{Hom}(U_i,A)|}{(G:U_i)}\right)^{m_i}.$$

We claim that terms with $m_\ell=0$ are small when compared to the whole sum. Since the number of summands is polynomial in n, it suffices to show that for every tuple $(m_1,\dots,m_{\ell-1},0)$ there exists a tuple $(m_1',\dots,m_{\ell-1}',m_\ell')$ with $m_\ell'\neq 0$, such that the summand corresponding to the first tuple is smaller than the one corresponding to the second by a factor $e^{cn^{1/d}}$. We do so by explicitly constructing the second tuple. Without loss we may assume that m_1 is maximal in the first tuple. We then set $m_1'=m_1-\lfloor cn^{1/d}\rfloor$, $m_\ell'=(G:U_1)\lfloor cn^{1/d}\rfloor$, and $m_i'=m_i$ for $i\neq 1,\ell$, where c is a positive constant chosen later. Then the product on the right hand side of the last displayed equation changes by a factor

$$\frac{m_1!}{(m_1-\lfloor cn^{1/d}\rfloor)!}\left(\frac{|A|^{(G:U_1)-1}|\operatorname{Hom}(U_1,A)|}{(G:U_1)|\operatorname{Hom}(G,A)|^{(G:U_1)}}\right)^{-\lfloor cn^{1/d}\rfloor)}\frac{1}{((G:U_1)\lfloor cn^{1/d}\rfloor)!}.$$

We may assume that n is sufficiently large, so that $m_1>2\lfloor cn^{1/d}\rfloor$. We can then estimate the factorials using the largest and smallest factors occurring. The other terms can be bounded rather careless to find that this quotient is at least

$$\left(\frac{m_1}{\left(cdn^{1/d}|A|\right)^d|\operatorname{Hom}(U_1,A)|}\right)^{\lfloor cn^{1/d}\rfloor}.$$

Since m_1 was chosen maximal we have $m_1\geq\frac{n}{|G|\ell}$, and we find that for $c^{-1}=ed|A|(\ell|\operatorname{Hom}(U_1,A))$ that the last expression is at least $e^{\lfloor cn^{1/d}\rfloor}$. Since c depends only on the subgroup U_1, we can take the minimum value over all the finitely many subgroups and obtain that there exists an absolute constant $c>0$, such that the number of homomorphisms φ such that $\pi\circ\varphi$ has no fixed point is by factor $\mathcal{O}(e^{-cn^{1/d}})$ smaller than the number of all homomorphisms. □

The theorem now follows from Lemma 1 and 3.

To deduce the corollary note that W_n is the subgroup of $C_2\wr S_n$ defined by the condition $(\pi;a_1,\dots,a_n)\in W_n\Leftrightarrow a_1+\dots+a_n=0$, that is, a homomorphism $\varphi:G\to C_2\wr S_n$ has image in W_n if and only if $\overline{\varphi}:G\to C_2$ is trivial. By the theorem the probability for this event differs from the probability that a random homomorphism $G\to C_2$ is trivial by $\mathcal{O}(e^{-cn^{1/d}})$, hence, we have

$$|\operatorname{Hom}(G, W_n)| = \left(\frac{1}{|\operatorname{Hom}(G, C_2)|} + \mathcal{O}(e^{-cn^{1/d}})\right) |\operatorname{Hom}(G, C_2 \wr S_n)|.$$

But there is a bijection between non-trivial homomorphisms $G \to C_2$ and subgroups of index 2, hence, $|\operatorname{Hom}(G, C_2)| = 1 + s_2(G)$, and the corollary follows.

References

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