Arch. Math. 74 (2000) 266–268
0003-889X/00/040266-03 $ 2.10/0

Archiv der Mathematik

On the class number of p-th cyclotomic field

By

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Abstract. Let $h^-(p)$ be the relative class number of the p-th cyclotomic field. We show that $\log h^-(p) = \frac{p+3}{4}\log p - \frac{p}{2}\log 2\pi + \log(1-\beta) + O(\log_2^2 p)$, where β denotes a Siegel zero, if such a zero exists and $p \equiv -1 \pmod 4$. Otherwise this term does not appear.

Assume that $D(s)$ is a Dirichlet series converging absolutely for $\Re s > 1$. Then it is a usual method to estimate $|D(1+it)|$ by estimating $|D(s)|$ trivially for $\Re s > 1$ and the modulus of its derivative or its logarithmic derivative by analytical means. In this note we will show that this method can be iterated, thus using not only the first derivative one can hope to obtain better bounds. The improvements achieved are significant only if in the halfplane $\Re s > 1$ substantially better estimates are available then to the left of 1. This is the case e.g. for the class number of the p-th cyclotomic field.

Let $k = \mathbb{Q}(\varepsilon)$ be the p-th cyclotomic field, $k^+ = k \cap \mathbb{R}$ its maximal real subfield. Let $h(p), h^+(p)$ be the class number of k, k^+ respectively, $h^-(p) = h(p)/h^+(p)$ the relative class number.

Theorem 1. *We have*

$$\log h^-(p) = \frac{p-3}{4}\log p - \frac{p}{2}\log 2\pi + \log(1-\beta) + O(\log_2^2 p)$$

where β is a Siegel zero of an L-series $\pmod p$, and this term does only occur, if such a zero is present and $p \equiv 3 \pmod 4$.

The proof will follow the lines of Masley and Montgomery [1]. As usual, let $\Pi(x,p,a) = \sum_{\substack{n \equiv a \pmod p \\ n \leqq x}} \frac{\Lambda(n)}{\log n}$.

Lemma 2. *If $x > p$ we have*

$$\Pi(x,p,a) \ll \frac{x}{(p-1)\log\left(\frac{x}{p}\right)}.$$

Mathematics Subject Classification (1991): 11R18, 11R29.

Proof. For $x > p \log p$ this can be deduced from the Brun-Titchmarsh inequality as in [1], Lemma 1. So we can assume $p < x < p^2$. Let q_1, q_2 be prime powers in this range, $q_i \equiv 1 \pmod{p}$. If q_1 and q_2 were different, but not coprime, then there quotient would be an integer $k \equiv 1 \pmod{p}$ with $2 < k < p$, which is clearly impossible. Thus if $N(x, Q, p, a)$ denotes the number of $n \equiv a \pmod{p}, n \leqq x$, such that n is not divisible by any prime number less than Q, we get $\Pi(x,p,a) < N(x,Q,p,a) + 1 + \pi(Q)\frac{\log Q}{\log p}$. The first term can be bounded using the large sieve inequality, the second e.g. by Tschebyscheffs inequality, thus $\Pi(x,p,a) \ll \frac{x}{p \log Q} + 1 + \frac{Q}{\log p}$. Now $Q = \frac{x}{p}$ gives the result.

Now define $f(s) = \sum\limits_{\chi(-1)=-1} \log L(s,\chi) - \log (s - \beta)$, where β is a Siegel zero, if such a zero occurs for some $L(s,\chi)$ with $\chi(-1) = -1$, especially if $p \equiv 1 \pmod{4}$ it does not occur at all. Then f is holomorphic and single valued in the union of the disc $|s-1| < \frac{c}{\log p}$ and the halfplane $\Re s > 1$. Choose the logarithm such that $f(s)$ is real for real s.

Lemma 3. *For $\sigma > 1$ we have*

$$|f(\sigma) + \log (1-\beta)| < \log \frac{1}{\sigma - 1} + c$$

and for $\nu \geqq 0$

$$|f^{(\nu+1)}(\sigma)| < \frac{c\nu!}{(\sigma-1)^{\nu}}.$$

Proof. This follows from Lemma 2 by partial summation.

Lemma 4 *In the rectangle $|\Im s| < \frac{1}{4}, \frac{3}{4} < \Re s < \frac{3}{2}$ we have*

$$\Re f(s) < p \log p + 2.$$

Proof. We have $L(s,\chi) = s \int\limits_1^\infty S(u) u^{-s-1} du$, where $S(u) = \sum\limits_{n \leqq u} \chi(n) \leqq \frac{p}{2}$. Thus $|L(s,\chi)| \leqq \frac{p \cdot |s|}{2\sigma} < p$ and $\Re \log L(s,\chi) < \log p$. By definition of a Siegel zero we can assume, that $\beta > \frac{9}{10}$. So we have $\Re f(s) < p \log p + 2$ on the boundary of the domain described, since $\Re f$ is harmonic with at most logarithmic singularities, in which $\Re f \to -\infty$, the same is true throughout the domain.

Lemma 5. *There is some $c > 0$, not depending on p, such that for $\sigma > 1$ we have*

$$|f^{(\nu+1)}(\sigma)| \ll p \log^{\nu+1} p c^{\nu} \nu!.$$

Proof. By Lemma 3 it suffices to consider the case $1 < \sigma < \frac{5}{4}$. For some $\delta > 0$, f is holomorphic in the union of the half strip $\Re s > 1, |\Im s| < \frac{\delta}{\log q}$ and the disc $|s-1| < \frac{\delta}{\log q}$. On any circle with radius $\frac{\delta}{\log q}$ and center $\sigma > 1$, $\Re f$ is bounded by Lemma 4, so by the Borel-Caratheodory lemma $|f'(s)| \leqq 8p \log^2 p \delta^{-1}$ for all s in the disc $|s-1| < \frac{\delta}{2 \log p}$ or the rectangle $1 < \Re s < \frac{5}{4}, |\Im s| < \dfrac{\delta}{2\log p}$. Now differentiating Cauchy's integral formula ν-times gives the result with $c = 2\delta^{-1}$.

Proof of Theorem 1. Define

$$M_\nu := \max_{\sigma>1} |f^{(\nu)}(\sigma)|.$$

Then we have for any $\sigma_0 > 1$

$$M_\nu \leqq (\sigma_0 - 1)M_{\nu+1} + \frac{c(\nu-1)!}{(\sigma_0-1)^\nu},$$

since for $\sigma > \sigma_0, f^{(\nu)}$ can be estimated with Lemma 3, and for $1 < \sigma < \sigma_0$ a bound for $f^{(\nu)}(\sigma_0)$ and $f^{(\nu+1)}$ suffices. With $\sigma_0 = \nu c^{1/\nu} M_{\nu+1}^{-1/\nu}$ this becomes

$$M_\nu \leqq c\nu M_{\nu+1}^{1-1/(\nu+1)}.$$

Iterating this formula we get

$$M_1 \leqq c^{\log n} e^{c\log^2 n} M_n^{1/n}.$$

With the estimate of Lemma 5 this becomes

$$M_1 < e^{c\log^2 n} n p^{1/n} \log^{1+1/n} p.$$

With $n = \log p$ we finally get $M_1 < e^{c\log_2^2 p}$, i.e. $|f'(\sigma)| < e^{c\log_2^2 p}$ for all $\sigma > 1$. Thus $|f(1)| < |f(\sigma)| + (\sigma-1)e^{c\log_2^2 p}$, with $\sigma = 1 + e^{-c\log_2^2 p}$ and using Lemma 3 we get $|f(1)| \ll \log_2^2 p$. Since as a special case of the class number formula (see e.g. [2]) we have the formula

$$\log h^-(p) = \frac{p-3}{4}\log p - \frac{p}{2}\log 2\pi + \log(1-\beta) + f(1)$$

the statement of the theorem follows.

References

[1] J. M. MASLEY and H. L. MONTGOMERY, Cyclotomic fields with unique factorization. J. Reine Angew. Math. **286/287,** 248–256 (1975).

[2] L. C. WASHINGTON, Introduction to cyclotomic fields. Graduate Texts in Math., **83**. New-York-Berlin (1982).

Eingegangen am 7. 12. 1998

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