# LOWER BOUNDS FOR EXPRESSIONS OF LARGE SIEVE TYPE

JAN-CHRISTOPH SCHLAGE-PUCHTA

Abstract. We show that the large sieve is optimal for almost all exponential sums.

Keywords: Large sieve, random series

MSC-Index: 11N35, 30B20

Let $a_n, 1 \leq n \leq N$ be complex numbers, and set $S(\alpha) = \sum_{n \leq N} a_n e(n\alpha)$, where $e(\alpha) = \exp(2\pi i\alpha)$. Large Sieve inequalities aim at bounding the number of places where this sum can be extraordinarily large, the basic one being the bound

$$\sum_{q \leq Q} \sum_{\substack{1 \leq a \leq q \\ (a,q)=1}} \left| S\left(\frac{a}{q}\right)\right|^2 \leq (N + Q^2) \sum_{n \leq N} |a_n|^2$$

(see e.g. [3] for variations and applications). P. Erdős and A. Rényi [1] considered lower bounds of the same type, in particular they showed that the bound

$$\sum_{q \leq Q} \sum_{(a,q)=1} \left| S\left(\frac{a}{q}\right)\right|^2 \ll N \sum_{n \leq N} |a_n|^2, \tag{1}$$

valid for $Q \ll \sqrt{N}$, is wrong for almost all choices of coefficients $a_n \in \{1, -1\}$, provided that $Q > C\sqrt{N} \log N$, and that the standard probabilistic argument fails to decide whether (1) is true in the range $\sqrt{N} < Q < \sqrt{N} \log N$. In this note, we show that (1) indeed fails throughout this range.

**Theorem 1.** *Let $S(\alpha)$ be as above. Then*

$$\sum_{q \leq Q} \sum_{(a,q)=1} \left| S\left(\frac{a}{q}\right)\right|^2 \geq \varepsilon Q^2 \sum_{n \leq N} |a_n|^2 \tag{2}$$

*holds true with probability tending to 1 provided $\varepsilon$ tends to 0, and $Q^2/N$ tends to infinity.*

Our approach differs from [1] in so far as we first prove an unconditional lower bound, which involves an awkward expression, and show then that almost always this expression is small. We show the following.

**Lemma 1.** *Let $S(\alpha)$ be as above, and define*

$$M(x) = \sup_{\mathfrak{m}} \frac{\int\limits_{\mathfrak{m}} |S(u)|^2 \, du}{\int\limits_0^1 |S(u)|^2 \, du},$$

*where* $\mathfrak{m}$ *ranges over all measurable subsets of* $[0,1]$ *of measure* $x$*. Then for any real parameter* $A>1$ *we have the estimate*

$$\sum_{q\leq Q}\sum_{(a,q)=1}\left|S\left(\frac{a}{q}\right)\right|^2 \geq \left(\frac{Q^2}{A}\left(1-M\left(\frac{1}{A}\right)\right)-6\pi NA\right)\sum_{n\leq N}|a_n|^2. \tag{3}$$

*Proof.* Our proof adapts Gallagher's proof of an upper bound large sieve [2]. For every $f\in C^1([0,1])$, we have

$$f(1/2)=\int_0^1 f(u)\,du+\int_0^{1/2} uf'(u)\,du-\int_{1/2}^1(1-u)f'(u)\,du.$$

Putting $f(u)=|S(u)|^2$, and using the linear substitution $u\mapsto(\alpha-\delta/2)+\delta u$, we obtain for every $\delta>0$ and any $\alpha\in[0,1]$

$$\begin{aligned}|S(\alpha)|^2 &= \frac{1}{\delta}\int_{\alpha-\delta/2}^{\alpha+\delta/2}|S(u)|^2\,du+\frac{1}{\delta}\int_{\alpha-\delta/2}^{\alpha}\big(\delta/2-|u-\alpha|\big)\big(S'(u)S(-u)-S(u)S'(-u)\big)\,du\\ &\quad -\frac{1}{\delta}\int_{\alpha}^{\alpha+\delta/2}\big(\delta/2-|u-\alpha|\big)\big(S'(u)S(-u)-S(u)S'(-u)\big)\,du.\end{aligned}$$

We have $|S(u)|=|S(-u)|$ and $|S'(-u)|=|S'(u)|$, thus $|S'(u)S(-u)-S(u)S'(-u)|\leq 2|S(u)S'(u)|$, and we obtain

$$\begin{aligned}|S(\alpha)|^2 &\geq \frac{1}{\delta}\int_{\alpha-\delta/2}^{\alpha+\delta/2}|S(u)|^2\,du-\frac{1}{\delta}\int_{\alpha-\delta/2}^{\alpha+\delta/2}2\Big(\frac{1}{2}-\frac{|u-\alpha|}{\delta}\Big)|S(u)S'(u)|\,du.\\ &\geq \frac{1}{\delta}\int_{\alpha-\delta/2}^{\alpha+\delta/2}|S(u)|^2\,du-\int_{\alpha-\delta/2}^{\alpha+\delta/2}|S(u)S'(u)|\,du.\end{aligned}$$

We now set $\delta=A/Q^2$. We can safely assume that $\delta<\frac{1}{2}$, since our claim would be trivial otherwise. Summing over all fractions $\alpha=\frac{a}{q}$ with $q\leq Q,(a,q)=1$, we get

$$\begin{aligned}\sum_{q\leq Q}\sum_{(a,q)=1}\left|S\left(\frac{a}{q}\right)\right|^2 &\geq \frac{Q^2}{A}\int_0^1|S(u)|^2\,du\,-\\ &\qquad\frac{Q^2}{A}\int_{m(Q,A)}|S(u)|^2\,du\;-\;\int_0^1 R(u)|S(u)S'(u)|\,du,\end{aligned} \tag{4}$$

where

$$R(u)=\#\left\{a,q:(a,q)=1,q\leq Q,\left|u-\frac{a}{q}\right|\leq\frac{A}{Q^2}\right\},$$

and

$$m(Q,A)=\{u\in[0,1]:R(u)=0\}.$$

To bound $R(u)$, let $\frac{a_1}{q_1} < \frac{a_2}{q_2} < \dots < \frac{a_k}{q_k}$ be the list of all fractions with $q_i \leq Q$, $\left|u - \frac{a_i}{q_i}\right| \leq \frac{A}{Q^2}$. We have for $i \neq j$ the bound

$$\left|\frac{a_i}{q_i} - \frac{a_j}{q_j}\right| \geq \frac{1}{q_i q_j} \geq \frac{1}{Q^2},$$

that is, the fractions $\frac{a_1}{q_1}, \dots, \frac{a_k}{q_k}$ form a set of points with distance $> \frac{1}{Q^2}$ in an interval of length $\frac{2A}{Q^2}$. There can be at most $2A+1$ such points, hence, $R(u) \leq 3A$.

Next, we bound $|m(Q,A)|$. By Dirichlet's theorem, we have that for each real number $\alpha \in [0,1]$ there exists some $q \leq Q$ and some $a$, such that $|\alpha - \frac{a}{q}| \leq \frac{1}{qQ}$. If $\alpha \in m(Q,A)$, we must have $\frac{1}{qQ} > \frac{A}{Q^2}$, that is, $q < Q/A$. Hence, we obtain

$$\begin{aligned}
|m(Q,A)| &\leq \left|\bigcup_{q<Q/A} \bigcup_{(a,q)=1} \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right] \setminus \left[\frac{a}{q} - \frac{A}{Q^2}, \frac{a}{q} + \frac{A}{Q^2}\right]\right| \\
&\leq \sum_{q<Q/A} \frac{\varphi(q)(2Q-2Aq)}{qQ^2} \leq \frac{1}{Q^2}\int_0^{Q/A} 2Q - 2At\, dt = \frac{1}{A}.
\end{aligned}$$

We can now estimate the right hand side of (4). The first summand is $\frac{Q^2}{A}\sum_{n\leq N}|a_n|^2$, while the second is by definition at most $\frac{Q^2}{A}M(1/A)$. For the third we apply the Cauchy-Schwarz-inequality to obtain

$$\begin{aligned}
\left(\int_0^1 |S(u)S'(u)|\, du\right)^2 &\leq \left(\int_0^1 |S(u)|^2\, du\right)\left(\int_0^1 |S'(u)|^2\, du\right) \\
&= \left(\sum_{n\leq N} |a_n^2|\right)\left(\sum_{n\leq N} (2\pi n)^2 |a_n^2|\right) \\
&\leq (2\pi N)^2 \left(\sum_{n\leq N} |a_n^2|\right)^2.
\end{aligned}$$

Hence, the last term in (4) is bounded above by $3A(2\pi N)\sum_{n\leq N}|a_n|^2$, and inserting our bounds into (4) yields the claim of our lemma. $\square$

Now we deduce Theorem 1. Let $S(\alpha)$ be a random sum in the sense that the coefficients $a_n \in \{1,-1\}$ are chosen at random. We compute the expectation of the fourth moment of $S(\alpha)$.

$$\begin{aligned}
\mathrm{E}\int_0^1 |S(u)|^4\, du &= \mathrm{E}\sum_{\substack{\mu_1+\mu_2=\nu_1+\nu_2 \\ \mu_1,\mu_2,\nu_1,\nu_2\leq N}} a_{\nu_1}a_{\nu_2}a_{\mu_1}a_{\mu_2} \\
&= \#\{\mu_1,\mu_2,\nu_1,\nu_2 \leq N : \{\mu_1,\mu_2\} = \{\nu_1,\nu_2\}\} \\
&= 2N^2 - N.
\end{aligned}$$

If $m \subseteq [0,1]$ is of measure $x$, then $\int_m |S(u)|^2\, du \leq \sqrt{x}\left(\int_m |S(u)|^4\, du\right)^{1/2}$, thus $\mathrm{E}M(x) \leq \sqrt{2x}$. In particular, we have $M(x) \leq 1/2$ with probability $\geq 1 - \sqrt{8x}$.

Let $\delta > 0$ be given, and set $A = 8\delta^{-2}$. Then with probability $\geq 1 - \delta$ we have $M(1/A) \leq 1/2$, and (3) becomes

$$\begin{aligned}\sum_{q\leq Q}\sum_{(a,q)=1}\left|S\left(\frac{a}{q}\right)\right|^2 &\geq \left(\frac{Q^2\delta^2}{16} - 48\delta^{-2}\pi N\right)\sum_{n\leq N}|a_n|^2\\ &\geq \frac{Q^2\delta^2}{32}\sum_{n\leq N}|a_n|^2,\end{aligned}$$

provided that $Q^2 > 1536\delta^4 N$. Hence, for fixed $\epsilon$, the relation (2) becomes true with probability $1 - \sqrt{1024\epsilon}$, provided that $Q^2/N$ is sufficiently large. Hence, our claim follows.

I would like to thank the referee for improving the quality of this paper.

## References

[1] P. Erdős, A. Rényi, Some remarks on the large sieve of Yu. V. Linnik, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **11** (1968), 3–13.

[2] P. X. Gallagher, The large sieve, *Mathematika* **14** (1967), 14–20.

[3] H. L. Montgomery, *Multiplicative Number Theory*, Lecture Notes in Mathematics, Vol. 227 (1971).

Jan-Christoph Schlage-Puchta
Department of Pure Mathematics and Computer Algebra
Universiteit Gent
Krijgslaan 281
9000 Gent
Belgium
jcsp@cage.ugent.be