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ON LARGE OSCILLATIONS OF THE REMAINDER OF THE PRIME NUMBER THEOREMS

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Abstract. Under the assumption of the appropriate Riemann hypothesis it is shown that $\max_{t \leqq x} \min_{l \neq 1} t^{-1/2}\big(\Psi(x,q,1) - \Psi(x,q,l)\big) > \big(\frac{1}{2} - \varepsilon\big) \log_3 x$ and $\min_{t \leqq x} \max_{l \neq 1} t^{-1/2}\big(\Psi(x,q,1) - \Psi(x,q,l)\big) < -\big(\frac{1}{2} - \varepsilon\big) \log_3 x$ for $x > x_0(q,\varepsilon)$. The proof is quite elementary, and x_0 can be estimated effectively. As a by-product a formula for the k-th power moment of certain normed error terms is obtained.

1. Introduction

Assume GRH. Then Wintner [9] has shown that $\Delta(t) = \frac{\Psi(e^t) - e^t}{e^{t/2}}$ has a distribution function with finite moments. More generally let q be a natural number, and consider

$$\Delta(t,q,l) = \frac{\Psi(e^t,q,l) - \frac{1}{\varphi(q)} e^t}{e^{t/2}}.$$

Then one can ask about the distribution of $f(t) = \big(\Delta(t,q,l_1), \dots, \Delta\big(t,q,l_{\varphi}(q)\big)\big)$, where l_i runs over a reduced systems of residues (mod q). A question of special interest is the so called Shanks–Rényi-race: given a permutation σ of the relative prime residue classes (mod q), is there a real number x such that $\pi\big(x,q,\sigma(1)\big) > \pi\big(x,q,\sigma(2)\big) > \dots > \pi\big(x,q,\sigma\big(\varphi(q)\big)\big)$? Assuming GRH, J. Kaczorowski has shown in [6] that for $q = 5$ and Ψ instead of π this is indeed true, and in [7] he showed that there are arbitrary large x such that $\pi(x,q,1) > \pi(x,q,a)$ for all $a \not\equiv 1 \pmod q$. Using a more elementary approach, we will prove a similar result which gives explicit estimates for $\pi(x,q,1) - \pi(x,q,a)$.

In this article we will always assume the Riemann hypothesis for all Dirichlet series occurring, and for every nontrivial zero ρ we set $\rho = \frac{1}{2} + i\gamma$. $\sum^*$ stands for summation restricted to those parameters described in the context.

Explicit bounds for the constants implied by Theorem 8, especially estimates for the first sign change of $\pi(x) - \operatorname{li} x$ and $\pi(x,q,1) - \pi(x,q,a)$ will be part of a subsequent paper.

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2. The moments of the error term

Assuming RH, Cramér [1] gave an explicit expression of the mean square of the normed error in the prime number formula. Wintner proved that the error term has k-th power moments for all $k \geqq 1$. The aim of this section is to compute these moments.

THEOREM 1. *Let χ be a charakter* (mod q), *k a natural number. Then*

$$\frac{1}{x}\int_0^x \left(\frac{\Psi(e^t,\chi)-Ee^t}{e^{t/2}}\right)^k dt \sim (-1)^k \sum_{\gamma_1+\ldots+\gamma_k=0} \frac{1}{\rho_1\cdots\rho_k}$$

where the sum runs over all nontrivial zeros of $L(s,\chi)$.

COROLLARY 2. *Assume that the positive imaginary parts of the zeros of ζ are linearly independent. Then all odd moments vanish, and the even moments can be expressed using power sums of zeros. In particular, we have*

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x \left(\frac{\Psi(e^s)-e^s}{e^{s/2}}\right)^2 ds = \sum_\rho \frac{1}{|\rho|^2},$$

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x \left(\frac{\Psi(e^s)-e^s}{e^{s/2}}\right)^4 ds = 2\cdot\left(\sum_\rho \frac{1}{|\rho|^2}\right)^2 - \sum_\rho \frac{1}{|\rho|^4},$$

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x \left(\frac{\Psi(e^s)-e^s}{e^{s/2}}\right)^6 ds$$

$$= 10\cdot\left(\sum_\rho \frac{1}{|\rho|^2}\right)^3 + 5\sum_\rho \frac{1}{|\rho|^2}\sum_\rho \frac{1}{|\rho|^4} - 24\sum_\rho \frac{1}{|\rho|^6}.$$

COROLLARY 3. *Assume RH. Then the third moment is $\leqq 0$.*

PROOF OF THEOREM 1. For $T > x^2$ we have

$$\Psi(x,\chi) = Ex - \sum_{|\rho|<T} \frac{x^\rho}{\rho} + O(\log x).$$

Since we assume $\Re\,\rho = 1/2$ for all nontrivial zeros, we get with $T > e^{2t}$

$$\left(\frac{\Psi(e^t) - Ee^t}{e^{t/2}}\right)^k = \left(-\sum_{|\rho|<T} \frac{e^{it\gamma}}{\rho} + O(te^{-t/2})\right)^k$$

$$= \left(-\sum_{|\rho|<T} \frac{e^{it\gamma}}{\rho}\right)^k + O(t^{2k-1}e^{-t/2})$$

$$= (-1)^k \sum_{|\rho_1|,\dots,|\rho_k|<T} \frac{e^{it(\gamma_1+\dots+\gamma_k)}}{\rho_1\cdots\rho_k} + O(t^{2k-1}e^{-t/2}).$$

Integrating this from 0 to x yields with $T = e^{2x}$

$$\int_0^x \left(\frac{\Psi(e^t) - Ee^t}{e^{t/2}}\right)^k dt$$

$$= (-1)^k \int_0^x \left\{ \sum_{|\rho_1|,\dots,|\rho_k|<T} \frac{e^{it(\gamma_1+\dots+\gamma_k)}}{\rho_1\cdots\rho_k} + O(t^{2k-1}e^{-t/2})\right\} dt$$

$$= (-1)^k \sum_{|\rho_1|,\dots,|\rho_k|<T} \int_0^x \frac{e^{it(\gamma_1+\dots+\gamma_k)}}{\rho_1\cdots\rho_k}\, dt + O(1)$$

$$= (-1)^k x \cdot \sum_{\substack{|\rho_1|,\dots,|\rho_k|<T \\ \gamma_1+\dots+\gamma_k=0}} \frac{1}{\rho_1\cdots\rho_k}$$

$$+O\left(1 + \sum_{\substack{|\rho_1|,\dots,|\rho_k|<T \\ \gamma_1+\dots+\gamma_k\neq 0}} \frac{1}{\rho_1\cdots\rho_k} \min\left(x, \frac{1}{|\gamma_1+\dots+\gamma_k|}\right)\right).$$

To prove the theorem it suffices to show that the error term is $o(x)$, and that the sum in the main term converges absolutely for $T \to \infty$. Both statements follow from the fact that the series

$$(1) \qquad \sum_{\rho_1,\dots,\rho_k} \frac{1}{\rho_1\cdots\rho_k} \min\left(1, \frac{1}{|\gamma_1+\dots+\gamma_k|}\right)$$

converges absolutely. First, the series of the main term is contained in this series, so convergence of this series implies that of the main term. Second, let $\varepsilon > 0$ and restrict the summation to those k-tuples occurring in the error term. Then there is a finite number of summands such that the sum of the remaining terms is $< \varepsilon$. Since every single term of the series in the error term is $O(1)$, the contribution of these finitely many terms is $< C_\varepsilon$, say. The contribution of the remaining terms is

$$\sum_{\rho_1,\dots,\rho_k}^{*} \frac{1}{|\rho_1\cdots\rho_k|}\min\left(x, \frac{1}{|\gamma_1+\ldots+\gamma_k|}\right)$$

$$\leqq x\cdot\sum_{\rho_1,\dots,\rho_k}^{*} \frac{1}{|\rho_1\cdots\rho_k|}\min\left(1, \frac{1}{|\gamma_1+\ldots+\gamma_k|}\right) < \varepsilon x.$$

Thus the error term is $< C_\varepsilon + \varepsilon x$. For $\varepsilon \to 0$ this becomes $o(x)$. Thus it suffices to consider (1).

Without loss we can restrict the summation to k-tuples with $|\gamma_{i+1}| \geqq |\gamma_i|$ for every $i \leqq k-1$. Consider those k-tuples with $|\gamma_1+\ldots+\gamma_k| \geqq |\rho_k|^{1/2}$ first. Here we have

$$\sum_{\rho_1,\dots,\rho_k}^{*} \frac{1}{|\rho_1\cdots\rho_k|}\min\left(1, \frac{1}{|\gamma_1+\ldots+\gamma_k|}\right) \leqq \sum_{\rho_1,\dots,\rho_k}^{*} \frac{1}{|\rho_1\cdots\rho_{k-1}\cdot\rho_k^{3/2}|}$$

$$\leqq \sum_{\rho_1,\dots,\rho_k} \frac{1}{|\rho_1\cdots\rho_k|^{1+1/2k}} = \left(\sum_{\rho}\frac{1}{|\rho|^{1+1/2k}}\right)^k < \infty,$$

since by well known zero density estimates $N(T,\chi) \ll T\log T$. Now consider those k-tuples with $|\gamma_1+\ldots+\gamma_k| < |\rho_k|^{1/2}$. For fixed $\gamma_1,\dots,\gamma_{k-1}$ with $|\gamma_1 + \ldots + \gamma_{k-1}| = s$ their number is $\ll s^{1/2}\log(s+2)$, and each single term is $< \frac{1}{|\rho_1\cdots\rho_{k-2}\cdot\rho_{k-1}^2|}$. Since $s \leqq (k-1)|\gamma_{k-1}|$, we get

$$\sum_{\rho_1,\dots,\rho_k}^{*} \frac{1}{|\rho_1\cdots\rho_k|}\min\left(1, \frac{1}{|\gamma_1+\ldots+\gamma_k|}\right) \ll \sum_{\rho_1,\dots,\rho_{k-1}} \frac{\log\big(|\rho_{k-1}|+2\big)}{|\rho_1\cdots\rho_{k-2}\rho_{k-1}^{3/2}|}$$

$$< \sum_{\rho_1,\dots,\rho_{k-1}} \frac{\log\big(|\rho_{k-1}+2\big)}{|\rho_1\cdots\rho_{k-2}\rho_{k-1}|^{1+1/2k}} < \left(\sum_{\rho}\frac{\log\big(|\rho|+2\big)}{|\rho|^{1+1/2k}}\right)^{k-1} < \infty. \qquad \square$$

PROOF OF COROLLARY 2. Due to Theorem 1 we have to determine all k-tuples of zeros of ζ with real sum. But since we assume the zeros to be linearly independent, such a k-tuple consists of $\frac{k}{2}$ pairs of conjugate roots. For odd k such k-tuples clearly cannot exist, and for even k an inclusion-exclusion argument yields the formulas given in Corollary 2. $\square$

PROOF OF COROLLARY 3. By Theorem 1 we have to evaluate

$$\sum_{\Re \rho_1+\rho_2+\rho_3=0} \frac{1}{\rho_1\rho_2\rho_3}.$$

Now assume that ρ_1, ρ_2, ρ_3 are zeros occurring in the sum. Without loss of generality we can assume that $|\rho_3| > |\rho_2| \geqq |\rho_1|$, and that $\gamma_3 > 0$ and $\gamma_1, \gamma_2 < 0$. Then

$$\Re\frac{1}{\rho_1\rho_2\rho_3} = \frac{\Re\overline{\rho_1\rho_2\rho_3}}{|\rho_1\rho_2\rho_3|} = \frac{\frac{1}{8} - \frac{1}{2}(\gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_3)}{|\rho_1\rho_2\rho_3|}.$$

The denominator of this expression is real and positive. Since $\gamma_3 = -\gamma_1 - \gamma_2$, the numerator becomes

$$\frac{1}{8} + \frac{1}{2}(\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2) = \frac{1}{8} + \frac{1}{4}(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) > 0.$$

Thus every single term has positive real part, and the third moment is negative or 0, depending on whether there are roots with $\gamma_1 + \gamma_2 + \gamma_3 = 0$ or not. □

The next statement will be applied in the next section, where it gives almost-periodicity results which are similar to those obtained in [2]–[5].

PROPOSITION 4. *Let $\Delta_T(t,\chi) = \sum_{|\rho|>T} \frac{e^{it\gamma}}{\rho}$, where ρ runs over all non-trivial zeros of $L(s,\chi)$. Assume that RH holds for $L(s,\chi)$. Then for all real numbers $a < b$ we have*

$$\int_a^b \left|\Delta_T(t,\chi)\right|^2 dt \ll (b - a + \log T)\frac{\log^2 T}{T},$$

where the constant implied by the symbol $\ll$ depends on χ.

PROOF. We have

$$\int_a^b \left|\Delta_T(t,\chi)\right|^2 dt = \int_a^b \left|\sum_{\rho>T} \frac{e^{it\gamma}}{\rho}\right|^2 dt$$

$$= \sum_{|\rho_1|,|\rho_2|>T} \frac{1}{\rho_1\overline{\rho_2}}\frac{1}{\gamma_1 - \gamma_2}\left(e^{i(\gamma_1-\gamma_2)b} - e^{i(\gamma_1-\gamma_2)a}\right)$$

$$< \sum_{|\rho_1|,|\rho_2|>T} \frac{1}{|\rho_1\rho_2|} \min\left(b - a, \frac{2}{\gamma_1 - \gamma_2}\right).$$

By symmetry we can assume that $0 < \gamma_1 \leqq \gamma_2$, since pairs of zeros with different signs certainly contribute less than $\frac{\log^2 T}{T}$. The contribution of zeros with $|\rho_1 - \rho_2| < 1$ is

$$\ll (b-a) \sum_{|\rho|>T} \frac{\log|\rho|}{\rho^2} \ll (b-a)\frac{\log^2 T}{T}.$$

Now let $n \geqq N$, $m \geqq 1$. Then the contribution of those zeros with $n \leqq |\rho_1| < n+1$, $m \leqq |\gamma_1 - \gamma_2| < m+1$ is $\ll \frac{\log n \log(n+m)}{nm(n+m)}$, and the contribution of the remainig zeros is at most

$$\sum_{n \geqq T} \sum_{m=1}^{\infty} \frac{\log n \log(n+m)}{nm(n+m)} \ll \sum_{n \geqq T} \sum_{m=1}^{n} \frac{\log^2 n}{n^2 m} + \sum_{m \geqq T} \sum_{n=T}^{m} \frac{\log n \log m}{nm^2}$$

$$\ll \sum_{n \geqq T} \frac{\log^3 n}{n^2} + \sum_{m \geqq T} \frac{\log^3 m}{m^2} \ll \frac{\log^3 T}{T}.$$

Thus the contribution of all zeros is

$$\ll (b-a)\frac{\log^2 T}{T} + \frac{\log^3 T}{T} = (b - a + \log T)\frac{\log^2 T}{T}. \qquad \square$$

3. Oscillation in the distribution of primes

The main result of this article is the following theorem.

THEOREM 5. *Let q be some natural number, $\varepsilon > 0$. Let*

$$\Delta(t,\chi) = \frac{\Psi(t,\chi) - Et}{\sqrt{t}},$$

where $E = 1$ resp. 0, depending on whether χ is principal or not. Assume RH for all L-series (mod q). *Then there are effective computable constants $X_0(q,\varepsilon)$ and $C = C(q)$ such that for all $X > X_0$ there are numbers $2 < x_+, x_- < X$ such that*

1. $\left|\Delta(x_+,\chi_i) - \Delta(x_+,\chi_j)\right| < C$,
2. $\Delta(x_+,\chi_i) > \left(\frac{1}{2} - \varepsilon\right) \log_3 X$,
3. $\left|\Delta(x_-,\chi_i) - \Delta(x_-,\chi_j)\right| < C$,
4. $\Delta(x_-,\chi_j) < -\left(\frac{1}{2} - \varepsilon\right) \log_3 X$,

5. $x_+ < x_- < x_+\left(1 + \frac{1}{\log_2^{1-\varepsilon} X}\right)$

where χ_i, χ_j *run over all characters* (mod q).

COROLLARY 6. *Assume RH. Then* $\Psi(x) = x + \Omega_\pm\left(\sqrt{x}\log_3 x\right)$, *and the implied constants are effectively computable.*

This was proven by Littlewood [8] in 1918, however, his estimate was not effective.

PROOF. Theorem 5 with $q = 1$ gives $\Delta(x,1) = \Omega_\pm(\log_3 x)$. Since $\Psi(x) - x = \sqrt{x}\Delta(x,1)$, we get $\Psi(x) - x = \Omega_\pm\left(\sqrt{x}\log_3 x\right)$ as claimed. □

COROLLARY 7. *Let* q *be an integer,* $\varepsilon > 0$ *and assume RH for all L-series* (mod q). *Then there is an effective computable constant* $X_0(q,\varepsilon)$ *such that for all* $X > X_0$ *there exist* $x_1, x_2 \in (0, X)$ *such that*

$$\min_{a \not\equiv 1 \pmod q} \Psi(x_1,q,1) - \Psi(x_1,q,a) > \left(\frac{1}{2} - \varepsilon\right)\sqrt{x_1}\log_3 X,$$

$$\max_{a \not\equiv 1 \pmod q} \Psi(x_2,q,1) - \Psi(x_2,q,a) < -\left(\frac{1}{2} - \varepsilon\right)\sqrt{x_2}\log_3 X.$$

J. Kaczorowski [7] proved Corollary 7 with $\omega(x)$ instead of $\log_3 x$, where $\omega(x) \nearrow \infty$ with $x \to \infty$.

PROOF. Let x_+ be the real number described in Theorem 5. Then we have

$$\Psi(x_+,q,1) = \frac{1}{\varphi(q)}\sum_\chi \Psi(x_+,\chi)$$

$$= \frac{x}{\varphi(q)} + \frac{\sqrt{x}}{\varphi(q)}\sum_\chi \Delta(x_+,\chi) > \frac{x}{\varphi(q)} + \left(\frac{1}{2} - \varepsilon\right)\sqrt{x}\log_3 X.$$

On the other hand for $a \not\equiv 1 \pmod q$ we have

$$\Psi(x_+,q,a) = \frac{1}{\varphi(q)}\sum_\chi \overline{\chi(a)}\Psi(x_+,\chi)$$

$$= \frac{x_+}{\varphi(q)} + \frac{\sqrt{x_+}}{\varphi(q)}\sum_\chi \overline{\chi(a)}\underbrace{\left(\Delta(x_+,\chi) - \Delta(x_+,\chi_0)\right)}_{\ll 1} + \underbrace{\frac{\sqrt{x_+}}{\varphi(q)}\sum_\chi \overline{\chi(a)}\Delta(x_+,\chi_0)}_{=0}$$

$$= \frac{x_+}{\varphi(q)} + O\left(\sqrt{x_+}\right).$$

Thus for $a \not\equiv 1$ we get

$$\Psi(x_+, q, 1) - \Psi(x_+, q, a) > \left(\frac{1}{2} - \varepsilon\right) \sqrt{x_+} \log_3 X - cx$$

for some constant c. Taking minimum over all $a \not\equiv 1 \pmod q$ we obtain the statement of the corollary by increasing ε and choosing X_0 large enough.

The proof of the second inequality is similar using x_- instead of x_+. □

The proof of Theorem 5 will partly depend on the following two lemmata.

LEMMA 8. *Let χ be a character* (mod q)*, and set* $\Delta(t, \chi) = \sum_\rho \frac{e^{it\gamma}}{\rho}$.

1. *If χ is real, we have*

$$f_1(t) := \Delta(t, \chi) + \Delta(-t, \chi) \ll 1,$$

and f_1 is continuous and piecewise differentiable on $\mathbf{R}$.

2. *If χ is complex, we have*

$$f_2(t) := \Delta(t, \chi) + \Delta(-t, \chi) + \Delta(t, \overline{\chi}) + \Delta(-t, \overline{\chi}) \ll 1,$$

and f_2 is continuous and piecewise differentiable on $\mathbf{R}$.

3. *If χ is complex, $t > 0$, we have*

$$f_3(t) := \Delta(t, \chi) + \Delta(-t, \chi) \ll 1 + \log^2\left(e^t + \frac{1}{t}\right) \min(1, t).$$

PROOF. The zeros of $L(s, \chi)$ are complex conjugate for χ real, and the zeros of $L(s, \chi)$ are conjugate zeros of $L(s, \overline{\chi})$ for χ complex, so f_1 and f_2 can be expressed as sums over pairs of conjugate terms. Now the contribution of such a pair of zeros to f_1 resp. f_2 is

$$\frac{e^{it\gamma} + e^{-it\gamma}}{\rho} + \frac{e^{it\gamma} + e^{-it\gamma}}{\bar{\rho}} = (e^{it\gamma} + e^{-it\gamma}) \frac{1}{|\rho|^2}.$$

Since $|e^{it\gamma} + e^{-it\gamma}| \leqq 2$, the sum over all zeros converges absolutely and is bounded above by $\sum_\rho \frac{2}{|\rho|^2}$. Thus f_1 and f_2 are both $\ll 1$ and continuous, since they are uniform limits of continuous functions. Thus it remains to prove differentiability. Let x be some real number and I an interval containing x such that I contains no number $n \log p$ where n is an integer and p is a prime. Then it suffices to prove that the series obtained by differentiating the series for f_1 resp. f_2 termwise converges uniformly within I to a continuous function. Thus in the following computation the first equality sign will be justified by the result. First we will consider f_1:

$$\frac{d}{dt}\big(f_1(t)+f_1(-t)\big) = \sum_\rho \frac{d}{dt}(e^{it\gamma}+e^{-it\gamma})\frac{1}{|\rho|^2} = \sum_\rho (ie^{it\gamma}-ie^{-it\gamma})\frac{\gamma}{|\rho|^2}$$

$$= -\sum_\rho (e^{it\gamma}-e^{-it\gamma})\frac{1}{\rho} + r_{1,\rho}(t) = -f_1(t)+f_1(-t)+r_2(t).$$

Here $r_{1,\rho}(t) \ll \frac{1}{|\rho|^2}$ and it is continuous, so $r_2(t)$ is continuous, too. But $f_1(t)$ is everywhere continuous except for $t = n\log p$, and the series for f_1 converges uniformly in every interval avoiding such numbers, so the claim is proven for real χ. For complex χ the same computation applies, since doubling the number of occurring terms does not influence convergence.

Now consider f_3. Denote ρ_n the n-th zero of $L(s,\chi)$ with positive imaginary part, ρ_{-n} the n-th zero with negative imaginary part. Using the well known density estimate we have $|\rho_n + \rho_{-n}| \ll 1$. Thus we get

$$\frac{e^{it\gamma_n}+e^{-it\gamma_n}}{\rho_n} + \frac{e^{it\gamma_{-n}}+e^{-it\gamma_{-n}}}{\rho_{-n}}$$

$$= (e^{it\gamma_n}+e^{-it\gamma_n})\frac{1}{|\rho_n|^2} - (e^{it\gamma_n}+e^{-it\gamma_n})\left(\frac{1}{\bar{\rho_n}} - \frac{1}{\rho_{-n}}\right)$$

$$+\frac{1}{\rho_{-n}}\big((e^{it\gamma_{-n}}+e^{-it\gamma_{-n}}) - (e^{it\gamma_n}+e^{-it\gamma_n})\big) \ll \frac{1}{|\rho_n|^2} + \frac{1}{|\rho_n|}\min(1,t)$$

since

$$|e^{-it\gamma_{-n}} - e^{it\gamma_n}| = |e^{it\gamma_n}|\cdot|e^{-it\gamma_{-n}-it\gamma_n} - 1|$$

$$< \min(t\gamma_{-n} - t\gamma_n, 2) \ll \min\big(\underbrace{(\gamma_n+\gamma_{-n})}_{\ll 1}t, 1\big) \ll \min(1,t).$$

Denote $g_3(t)$ the finite series $\sum\limits_{|\rho|<T} \frac{e^{it\gamma}+e^{-it\gamma}}{\rho}$, where T will be determined later. By the known estimate for the truncation error in the explicit formula for $\Psi(x,\chi)$ we get

$$\big|f_3(t) - g_3(t)\big| \ll \frac{\log^2 T}{T}\left(e^t + \frac{1}{t^2}\right),$$

thus for $T = e^{2t} + \frac{1}{t^3}$ we can replace f_3 by g_3. Thus we get

$$g_3(t) + g_3(-t) \ll \sum_{|\rho|<e^{2t}+\frac{1}{t^3}} \frac{1}{|\rho|^2} + \frac{1}{|\rho|}\min(1,t)$$

$$= \sum_{|\rho|<e^{2t}+\frac{1}{t^3}} \frac{1}{|\rho|^2} + \sum_{|\rho|<e^{2t}+\frac{1}{t^3}} \frac{1}{|\rho|}\min(1,t).$$

The first sum can be bounded independently of t, for the second we use the inequality $\sum\limits_{|\rho|<T} \frac{1}{|\rho|} \ll \log^2 T$. Thus we obtain

$$g_3(t) + g_3(-t) \ll 1 + \sum_{\rho<e^{2t}+\frac{1}{t^3}} \frac{1}{|\rho_n|}\min(1,t)$$

$$\ll 1 + \log^2\left(e^{2t} + \frac{1}{t^3}\right)\min(1,t) \ll 1 + \log^2\left(e^{t} + \frac{1}{t}\right)\min(1,t)$$

which proves our claim. □

For real numbers $x_1, \dots, x_n$, let $\left\|(x_1, \dots, x_n)\right\|^2 = \sum\limits_{j=1}^{n} \{x_i\}^2$, where $\{x_i\}$ is the fractional part of x_i.

LEMMA 9. *Let n be a natural number, $\vec{\alpha} = (t_1, \dots, t_n) \in \mathbf{R}^n$, $\varepsilon > 0$. Then there is some s with $1 < s < \frac{2^n\Gamma(n/2)}{\pi^{n/2}\varepsilon^n} + 1 =: M + 1$ such that*

$$\left\| s \cdot (t_1, \dots, t_n)\right\| < \varepsilon.$$

PROOF. The proof will use the pigeon-hole-principle. For any integer k with $1 \leqq k \leqq M + 1$ set $x_k := k \cdot \vec{\alpha}$ and consider the balls with radius $\varepsilon/2$ and center x_k. If none of these intersect (mod 1) nontrivially, their volume is bounded by the volume of the unit cube, thus

$$\omega_n \cdot (\varepsilon/2)^n \cdot (M+1) < 1$$

where $\omega_n = \frac{\pi^{n/2}}{\Gamma(n/2)}$ is the volume of the n-dimensional unit sphere. By definition of M we obtain

$$\omega_n \cdot \frac{\Gamma(n/2)}{\pi^{n/2}} < 1, \qquad \omega_n < \frac{\pi^{n/2}}{\Gamma(n/2)}.$$

However, this contradicts the known formula for ω_n.

So the balls intersect in some point P, i.e. there are natural numbers $k_1 < k_2 < M$ such that $P \in B_{x_{k_1}}(\varepsilon/2) \cap B_{x_{k_2}}(\varepsilon/2)$. Hence

$$\|x_{k_2} - x_{k_1}\| \leqq \|x_{k_1} - P\| + \|P - x_{k_2}\| < \varepsilon,$$

but since $x_{k_2} - x_{k_1} \equiv x_{k_2-k_1} \pmod 1$, we get $\|x_{k_2-k_1}\| < \varepsilon$. Thus $s = k_2 - k_1$ has the claimed properties. □

PROOF OF THEOREM 5. Consider the explicit formula for $\Psi(x,\chi)$ in the version

$$\Psi(x,\chi) = Ex - \sum_\rho \frac{x^\rho}{\rho} + O(\log x) = Ex - \sqrt{x}g(\log x,\chi) + O(\log x).$$

To find large oscillation of Ψ, it suffices to find oscillation of g, more precisely we have to prove the following statement:

Let q be a natural number, $\varepsilon > 0$, and assume that no L-series (mod q) *vanishes in the region $\Re s > 1/2$. Then there are effectively computable constants $Y_0(q,\varepsilon)$ and C such that for all $Y > Y_0$ there are numbers $2 < s_+, s_- < Y$ such that*

1. $\left|g(s_+,\chi_i) - g(s_+,\chi_j)\right| < C,$
2. $g(s_+,\chi_j) > \left(\frac{1}{2} - \varepsilon\right) \log_2 Y,$
3. $\left|g(s_-,\chi_i) - g(s_-,\chi_j)\right| < C,$
4. $g(s_-,\chi_j) < -\left(\frac{1}{2} - \varepsilon\right) \log_2 Y,$
5. $s_+ < s_- < s_+ + \frac{1}{\log^{1-\varepsilon} Y}$

where χ_i and χ_j run over all characters (mod q).

Indeed, if we set $Y = \log X$, $t_+ = e^{s_+}$ and $t_- = e^{s_-}$, we obtain the statement of the theorem. Therefore in the sequel we will only consider the functions $g(t,\chi)$.

In the formula

$$\Psi(e^y,\chi) = E_\chi e^y - e^{\frac{1}{2}y}g(y,\chi) - \frac{1}{2}\log(1 - e^{-2y}) + O(1+y)$$

all terms are bounded for $y \searrow 0$, except $g(y,\chi)$ and $\frac{1}{2}\log(1-e^{-2y})$. Since $(e^{\frac{1}{2}y} - 1)\log(1-e^{-2y}) \to 0$ for $y \searrow 0$, the error of replacing $e^{\frac{1}{2}y}g(y,\chi)$ by $g(y,\chi)$ is $\ll 1$. Thus for $y > 0$ sufficiently small we get

$$g(y,\chi_j) = -e^{-\frac{1}{2}y}\frac{1}{2}\log\left(1 - e^{-2y}\right) + O(1) = -\frac{1}{2}\log\left(1 - e^{-2y}\right) + O(1)$$
$$= -\frac{1}{2}\log y + O(1) = \frac{1}{2}\log\frac{1}{y} + O(1),$$

i.e.

(2) $$g_j(y) > \left(\frac{1}{2} - \varepsilon\right) \log \frac{1}{y}.$$

On the other hand $\left|g_i(y) - g_j(y)\right| \ll 1$. Using Lemma 8 we get

(3) $$g_j(y) < -\left(\frac{1}{2} - 2\varepsilon\right) \log \frac{1}{|y|},$$

$\left|g_i(y) - g_j(y)\right| \ll 1$ for $y < 0$, sufficiently close to 0.

Now define $g(t) : \mathbf{R} \to \mathbf{R}^{\varphi(q)} : t \mapsto \left(g(t,\chi_1), \dots, g\left(t, \chi_{\varphi(q)}\right)\right)$ and $g_N(t) : \mathbf{R} \to \mathbf{R}^{\varphi(q)} : t \mapsto \left(g_N(t,\chi_1), \dots, g_N\left(t, \chi_{\varphi(q)}\right)\right)$, where $g_N(t,\chi)$ is the series restricted to the N zeros with least absolute values.

Using Proposition 4 we have, using $N(T,\chi) \ll T \log T$,

$$\int_x^{x+1} \left\| g(t) - g_N(t) \right\|^2 dt \ll \frac{\log^4 N}{N}.$$

Now if $N = \log^{1-\varepsilon} Y$, and Y is sufficiently large, the right hand side becomes $< \varepsilon^3 \log^{-1+2\varepsilon} Y$. Together with the estimate (2) we get that for all $0 < y < \log^{-1+3\varepsilon} Y$ with the possible exception of a set of measure $\varepsilon \log^{-1+2\varepsilon} Y$ at most we have for all $\chi \pmod q$ the estimate

$$g_N(y,\chi) > g(y,\chi) - \varepsilon > \left(\frac{1}{2} - 4\varepsilon\right) \log_2 Y.$$

Similarily, using (3) we get for $-\log^{-1+3\varepsilon} Y < y < 0$ with the possible exception of a set of measure $\varepsilon \log^{-1+2\varepsilon} Y$ at most the estimate

$$g_N(y,\chi) < g(y,\chi) + \varepsilon < -\left(\frac{1}{2} - 4\varepsilon\right) \log_2 Y.$$

Now we apply Lemma 9 with $n = kN$, $k = \varphi(q)$ and $\varepsilon = \frac{1}{4\pi\sqrt{kN}}$. We obtain the existence of a real number s with

$$1 \leqq s \leqq \frac{\left(8\pi\sqrt{kN}\right)^{kN} \Gamma(kN/2)}{\pi^{kN/2}} < e^{2kN \log kN} < e^{2k \log^{1-\varepsilon/2} Y} < Y - 1,$$

such that

$$\left(\sum_\rho \left\{\frac{s\gamma}{2\pi}\right\}\right)^2 \leqq kN \sum_\rho \left\{\frac{s\gamma}{2\pi}\right\}^2 \leqq \left(\frac{1}{4\pi}\right)^2$$

where ρ runs over the N zeros with least imaginary part of every L-series (mod q). Hence we conclude

$$\left\| g_N(y) - g_N(y+s) \right\|^2 \leqq \left(\sum_{j=1}^{k} \left| g_{Nj}(y) - g_{Nj}(y+s) \right| \right)^2$$

$$\leqq \left(\sum_\rho \left| \frac{e^{iy\gamma}}{\rho} - \frac{e^{i(y+s)\gamma}}{\rho} \right| \right)^2 \leqq \left(\sum_\rho \left| \frac{e^{is\gamma}-1}{\rho} \right| \right)^2$$

$$\leqq \left(2 \cdot 2\pi \sum_\rho \left\{ \frac{s\gamma}{2\pi} \right\} \right)^2 \leqq 1.$$

Now choosing $|y| < \log^{-1+3\varepsilon} Y$ and using the estimates for $g_N(y,\chi)$ given above, we conclude that in the interval $[s, s+\log^{-1+3\varepsilon} Y]$ with the possible exception of a set of measure $2\varepsilon \log^{-1+2\varepsilon} Y$ at most for all characters χ (mod q) the estimate

$$g_{Nj}(z) > \left(\frac{1}{2} - 4\varepsilon \right) \log_2 Y - 1 > \left(\frac{1}{2} - 5\varepsilon \right) \log_2 Y$$

holds, where $z = y + s$. In the same way we get for $y \in [s - \log^{-1+3\varepsilon} Y, s]$ with the possible exception of a set of measure $2\varepsilon \log^{-1+2\varepsilon} Y$ at most the inequality

$$g_{Nj}(z) < -\left(\frac{1}{2} - 4\varepsilon \right) \log_2 Y + 1 < -\left(\frac{1}{2} - 5\varepsilon \right) \log_2 Y.$$

Now using Proposition 4 once again we get that for $z \in [s, s+\log^{-1+3\varepsilon} Y]$ with the possible exception of a set of measure $\leqq 3\varepsilon \log^{-1+2\varepsilon} Y$ at most all the $\varphi(q)$ inequalities

$$g(z,\chi) > \left(\frac{1}{2} - 8\varepsilon \right) \log_2 Y$$

are valid. For $z \in [s - \log^{-1+3\varepsilon} Y, s]$ with the possible exception of a set of measure $\leqq 3\varepsilon \log^{-1+2\varepsilon} Y$ at most we have

$$g(z,\chi) < -\left(\frac{1}{2} - 8\varepsilon \right) \log_2 Y.$$

In the same way we obtain that in $[s - \log^{-1+3\varepsilon} Y, s + \log^{-1+3\varepsilon} Y]$ with the possible exception of a set of measure $3\varepsilon \log^{-1+2\varepsilon} Y$ at most all the inequalities $\left|g(z,\chi_i) - g(z,\chi_j)\right| \ll 1$ are valid.

Combining these considerations we get that the measure of $z \in [s, s + \log^{-1+3\varepsilon} Y]$ such that

$$g(z,\chi) > \left(\frac{1}{2} - 8\varepsilon\right) \log_2 Y$$

for all $\chi \pmod q$ and

$$\left|g(z,\chi_i) - g(z,\chi_j)\right| < C$$

for all $\chi_i, \chi_j \pmod q$ is at least $\log^{-1+3\varepsilon} Y - 6\varepsilon \log^{-1+2\varepsilon} Y$. Without loss of generality we can assume $\varepsilon < 1/6$, so this set is not empty. Let s_- be an arbitrary point from this set. Similarily there is an $s_+ \in [s - \log^{-1+3\varepsilon} Y, s]$, such that the inequalities

$$g(s_+,\chi) < -\left(\frac{1}{2} - 8\varepsilon\right) \log_2 Y$$

and

$$\left|g(s_+,\chi_i) - g(s_+,\chi_j)\right| < C$$

hold. Obviously we have $s_+ < s_- < s_+ + 2\log^{-1+3\varepsilon} Y < s_+ + \log^{-1+4\varepsilon} Y$.

Now replacing ε by $\varepsilon/8$, we obtain the claimed inequalities for g, and these imply the statement of our theorem. □

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