An inequality for means with applications

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Abstract. We show that an almost trivial inequality between the first and second moment and the maximal value of a random variable can be used to slightly improve deep theorems.

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One often estimates individual values of a function by computing a certain means. This approach is particularly useful in situations where the conjectured maximum of a function is close to its mean, as is often the case in number theory. Here, we give a simple method which sometimes allows to improve the resulting estimates. To ease aplications, we formulate our result in the language of probability theory.

Theorem 1. *Let ξ be a non-negative real random variable, and suppose that $\mathbf{E}\xi = 1$, and $\mathbf{E}\xi^2 = a$ with $a > 1$. Then the probability $P(\xi \geq a)$ is positive, and for every $b < a$ we have*

$$\int\limits_{|\xi|>b} \xi^2 \geq a - b.$$

Proof. We have

$$\int\limits_{|\xi|\leq b} \xi^2 \leq b \int\limits_{|\xi|>b} \xi \leq b \int \xi = b,$$

which implies the claimed inequality. Now suppose that $|\xi| < a$ almost surely. Then there exists some $\epsilon > 0$, such that $\int\limits_{|\xi|\leq a-\epsilon} \xi \geq \frac{1}{2}$, and we obtain

$$a = \int \xi^2 = \int\limits_{|\xi|\leq a-\epsilon} \xi^2 + \int\limits_{|\xi|>a-\epsilon} \xi^2$$

$$\leq (a-\epsilon) \int\limits_{|\xi|\leq a-\epsilon} \xi + a \int\limits_{|\xi|>a-\epsilon} \xi \leq \frac{(a-\epsilon)+a}{2} < a,$$

a contradiction. □

We now give three applications to quite different areas.

Our first application shows that the fourth moment of the Riemann ζ-function is dominated by large values of ζ, in fact, by values which are so large that the fourth moment itself cannot guarantee them to exist.

Corollary 1. *There is a constant C such that for $t > T_0(\epsilon)$ and $H > T^{2/3}\log^C T$ we have*

$$\int\limits_{\{t\in[T,T+H]:|\zeta(\frac{1}{2}+it)|>\frac{1}{4\pi^2}\log^{3/2} t\}} |\zeta(\frac{1}{2}+it)|^4\, dt \geq \frac{1-\epsilon}{4\pi^2} T\log^4 T + \mathcal{O}(T\log^3 T).$$

Proof. Ingham proved

$$\int_0^T |\zeta(\frac{1}{2}+it)|^2\, dt = T\log\frac{T}{2\pi} + (2\gamma-1)T + \mathcal{O}/T^{1/2+\epsilon}),$$

and Ivic and Motohashi[2] showed that

$$\int_0^T |\zeta(\frac{1}{2}+it)|^4\, dt = T\cdot P(\log T) + \mathcal{O}(T^{2/3}\log^C T),$$

where P is a polynomial of degree 4 with leading term $\frac{1}{2\pi^2}$ (confer also [1]). Now apply Theorem 1 by setting

$$\xi = H\cdot|\zeta(\frac{1}{2}+it)|^2 \left(\int\limits_T^{T+H} |\zeta(\frac{1}{2}+it)|^2\, dt\right)^{-1}$$

for $t \in [T, T+H]$ chosen at random and $b = \frac{1}{4\pi^2}\log^2 T$. □

Our second application slightly improves on the approach of Szekeres and Turán [5] on the problem of Hadamard-matrices.

Corollary 2. *For every $\epsilon > 0$ and $n > n_0(\epsilon)$ there exists a skew-symmetric $n\times n$-matrix A with entries ± 1 satisfying*

$$|\det A| > \left(\frac{n}{64\pi e^5}\right)^{1/4} e^{\sqrt{n}}\sqrt{n!}.$$

Proof. Let A be a random skew-symmetric $n \times n$-matrix with entries ± 1, and let $s_k(n)$ be the k-th mean of the determinant of A. Szekeres[4] showed that

$$\begin{aligned} s_1(n) &\sim \frac{1}{\sqrt[4]{8\pi e n}} e^{\sqrt{n}} \sqrt{n!}, \\ s_2(n) &\sim \frac{1}{\sqrt{32\pi e^3}} e^{2\sqrt{n}} \sqrt{n!}. \end{aligned}$$

Our claim now follows by applying our theorem to $\frac{\det A}{s_1(n)}$. □

Our last result improves on the work of Kerov and Vershik [3] concerning the largest degree of an irreducible character of the symmetric group.

Corollary 3. *Let $\epsilon > 0$ be given. Then for every $n > n_0(\epsilon)$ there exists an irreducible character χ of S_n with $\chi(1) > (1-\epsilon)e^{1/4}\sqrt{\pi n}e^{-\sqrt{n}}\sqrt{n!}$.*

Proof. All irreducible complex representations of S_n can be realized over $\mathbb{R}$, thus

$$\sum_\chi \chi(1) = \#\{\pi \in S_n : \pi^2 = \mathrm{id}\} \sim \frac{e^{\sqrt{n}-\frac{1}{4}}}{2\sqrt{\pi n}} \sqrt{n!},$$

whereas the orthogonality relation implies $\sum_\chi \chi(1)^2 = n!$. Finally, the number of irreducible characters equals the number $p(n)$ of partitions of n, for which we have the asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}.$$

Define a random variable ξ as $\frac{\chi(1)}{\sqrt{n!}}$, where χ is chosen at random among all irreducible characters, where each character has the same probability. Then we obtain

$$\begin{aligned} \mathbf{E}\xi &\sim \frac{2\sqrt{3n}}{e^{1/4}\sqrt{\pi}} \exp\big((1-\pi\sqrt{2/3})\sqrt{n}\big) \\ \mathbf{E}\xi^2 &\sim 4n\sqrt{3} \exp\big(-\pi\sqrt{2n/3}\big), \end{aligned}$$

and our claim follows. □

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