Analysis 25, 281–296 (2005)

Generalized partition functions and subgroup growth of free products of nilpotent groups

Thomas W. Müller, Jan-Christoph Schlage-Puchta

Received: August 2, 2005

Summary: We estimate the growth of homomorphism numbers of a torsion-free nilpotent group G using a variant of the circle method together with the analytic continuation of $\zeta_G(s)$ established in [4]. As an application, we obtain information on the subgroup growth of free products of nilpotent groups.

1 Introduction

Let G be a finitely generated torsion-free nilpotent group, and let $h_n(G) := |\mathrm{Hom}(G, S_n)| /n!$. In Section 2 we derive the infinite product generating function

$$H(z) := \sum_{n=0}^{\infty} h_n(G) z^n = \prod_{n=1}^{\infty} \left(1 - z^n\right)^{-\eta(n)},$$

where the exponents $\eta(n)$ can be expressed in terms of the numbers $s_n(G)$ of index n subgroups in G. This representation for $H(z)$ allows us to employ a variant of the circle method, originally developed in connection with partition generating functions, in order to obtain information on the asymptotics of $h_n(G)$, which is done in Section 3; the reader is referred to [1, Chapter 6] for a leisurely exposition of these techniques, as well as relevant references. Our analysis requires information on the Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} \eta(n)\, n^{-s},$$

which we derive from the proof of the main result in [4]; cf. Proposition 3.1. In this way we find in particular that

$$\log h_n(G) = \left(C + \mathcal{O}\left(\frac{1}{\log n}\right)\right) n^{(\alpha-1)/\alpha} \log^{d/\alpha} n;$$

cf. Theorem 3.3 for a more precise statement. Moreover, we show that, for G free abelian, the error term in the last equation can be replaced by an asymptotic series in negative

AMS 2000 subject classifications: Primary: 20E07; Secondary: 11P82, 11M41
Key words and phrases: subgroup growth, free products, generalized partition functions, circle method, nilpotent groups

powers of n; see Proposition 3.4. The final section employs these estimates on $h_n(G)$ to derive information concerning the subgroup growth of free products Γ of torsion-free nilpotent groups. Here, the transition from $h_n(\Gamma)$ to $s_n(\Gamma)$ is made possible by a result in [8] to the effect that almost all permutation representations of a free product are in fact transitive.

2 Infinite product generating functions associated with nilpotent groups

The infinite product representation provided by our first result will allow us to utilize analytic methods from the theory of partitions.

Lemma 2.1 *Let G be a finitely generated group. Then we have, as an equation between formal power series, the relation*

$$H(z) := \sum_{n=0}^{\infty} h_n(G) z^n = \prod_{n=1}^{\infty} (1 - z^n)^{-\eta(n)}, \tag{2.1}$$

where

$$\eta(n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) s_d(G). \tag{2.2}$$

In particular, $\eta(n)$ is multiplicative if and only if $s_n(G)$ is multiplicative, and $\eta(n)$ is positive for all n, provided that G is torsion-free, nilpotent, and not cyclic.

Proof: Existence and uniqueness of the product representation (2.1) is clear; hence it suffices to show that the numbers $\eta(n)$ given by equation (2.2) satisfy this relation. We have[1]

$$H(z) = \exp\left(\sum_{m=1}^{\infty} \frac{s_m(G)}{m} z^m\right). \tag{2.3}$$

Using the Möbius inversion formula and the Taylor series of $\log(1 - z)$, we obtain

$$\begin{aligned} H(z) &= \exp\left(\sum_{m=1}^{\infty} \sum_{d|m} \left(\frac{d\eta(d)}{m}\right) z^m\right) \\ &= \exp\left(\sum_{n=1}^{\infty} \sum_{\nu=1}^{\infty} \eta(n) \frac{z^{n\nu}}{\nu}\right) \\ &= \exp\left(-\sum_{n=1}^{\infty} \eta(n) \log(1 - z^n)\right) \\ &= \prod_{n=1}^{\infty} (1 - z^n)^{-\eta(n)}. \end{aligned}$$

[1] Cf. [10], or [3, Proposition 1].

Since $\eta(n)$ is the product of the multiplicative function $\frac{1}{n}$ and the Dirichlet convolution of the multiplicative function μ with $s_n(G)$, $\eta(n)$ is multiplicative if and only if $s_n(G)$ is multiplicative. If G is nilpotent, $s_n(G)$ and hence $\eta(n)$ is multiplicative, and $\eta(n)$ is positive if and only if this sequence is positive on prime powers. We have $\eta(p^k) = (s_{p^k}(G) - s_{p^{k-1}}(G))\, p^{-k}$, hence, $\eta(n)$ is positive if and only if $s_{p^k}(G)$ is strictly increasing in k for all primes p. Our last claim follows now from the next lemma. □

Lemma 2.2 *Let G be a finitely generated torsion-free nilpotent group, and let p be a prime. Suppose that G is not cyclic. Then we have*

$$\frac{s_{p^{k+1}}(G)}{s_{p^k}(G)} \geq \beta(p),$$

where $\beta(p)$ is independent of G, and satisfies $\beta(p) \geq \frac{4}{3}$ as well as $\beta(p) = p + \mathcal{O}(1)$.

Proof: We prove our claim by induction on the Hirsch length h of G. Set

$$\beta(p, h) := \inf_{\substack{h(G)=h \\ k\geq 0}} \frac{s_{p^{k+1}}(G)}{s_{p^k}(G)}.$$

The case $h = 1$ is excluded, and for $h = 2$ the only possible group is $\mathbb{Z}^2$, hence,

$$\beta(p, 2) = \inf_{k\geq 0} \frac{\sigma\left(p^{k+1}\right)}{\sigma\left(p^k\right)} = p.$$

For groups of Hirsch length 3, the ζ-functions are computed in [5, Proposition 8.1], and one finds after some computation that $\zeta_G(s) = \zeta(s-1)D(s)$, where $D(s) = \sum \frac{a_n}{n^s}$ is a Dirichlet-series with positive coefficients. Hence,

$$s_{p^{k+1}}(G) = \sum_{\kappa=0}^{k+1} p^\kappa a_{p^{k-\kappa+1}} = p\sum_{\mu=0}^{k} p^\mu a_{p^{k-\mu}} + a_{p^{k+1}} \geq p s_{p^k}(G),$$

hence, $\beta(p, 3) \geq p$.

For groups of Hirsch length ≥ 4, we use the concept of good bases as introduced in [5]; cf. also [7, Chapter 15.1]. Choose a Mal'cev basis $x_1, \dots, x_h$ of G such that $x_h \in Z(G)$, and define the subgroups $H_i = \langle x_i, \dots, x_h\rangle$. For a subgroup U, choose a good basis $a_1, \dots, a_h$, and define

$$s_{k,\nu}(G) := \left|\left\{U \leq G : (G:U) = p^k, (\langle a_h\rangle : \langle a_1, \dots, a_{h-1}\rangle \cap \langle a_h\rangle) = p^\nu\right\}\right|.$$

To show that $s_{k,\nu}(G)$ is well defined, we have to prove that the index

$$(\langle a_h\rangle : \langle a_1, \dots, a_{h-1}\rangle \cap \langle a_h\rangle)$$

does not depend on the good basis chosen for U. We have $\langle a_h\rangle = U\cap\langle x_h\rangle$. Let $a_1',\dots,a_h'$ be another good basis for U, and set $N=\langle x_h\rangle$. Then $a_i'N\in\langle a_1,\dots,a_{h-1}\rangle N$; hence, there exist p-adic integers $y_1,\dots,y_{h-1}$, such that

$$\left\langle a_1'x_h^{y_1}, a_2'x_h^{y_2},\dots,a_{h-1}'x_h^{y_{h-1}}\right\rangle \leq \langle a_1,a_2,\dots,a_{h-1}\rangle.$$

Since x_h is central, and G is torsion-free, we have

$$\left\langle a_1'x_h^{y_1}, a_2'x_h^{y_2},\dots,a_{h-1}'x_h^{y_{h-1}}\right\rangle \cap \langle x_h\rangle = \left\langle a_1',a_2',\dots,a_{h-1}'\right\rangle\cap\langle x_h\rangle,$$

and therefore

$$\left(\left\langle a_h'\right\rangle : \left\langle a_1',\dots,a_{h-1}'\right\rangle\cap\left\langle a_h'\right\rangle\right) \leq \left(\langle a_h\rangle : \langle a_1,\dots,a_{h-1}\rangle\cap\langle a_h\rangle\right).$$

By symmetry, we see that the two indices are equal; hence, $s_{k,\nu}(G)$ is indeed well defined.

Let U be a subgroup counted by $s_{k,\nu}(G)$ with $\nu\geq 1$. Consider the set of all tuples $b_1,\dots,b_h$, such that $b_i\in H_i\setminus H_{i+1}$ with $b_h=a_h^p$, and $b_1,\dots,b_{h-1},a_h$ is a good basis of U. Then $b_1,\dots,b_h$ is a good basis of a subgroup U' of index p^{k+1}, the measure of all good bases producing the same subgroup is the measure of all good bases producing U times p^{1-h}, and different U yield different subgroups U', since p-th roots in $\langle x_h\rangle$ are unique. Moreover, a subgroup U' produced in this way is counted by $s_{k+1,\nu-1}(G)$, since

$$\left(\langle b_h\rangle : \langle b_1,\dots,b_{h-1}\rangle\cap\langle b_h\rangle\right) = \left(\left\langle a_h^p\right\rangle : \langle a_1,\dots,a_{h-1}\rangle\cap\left\langle a_h^p\right\rangle\right) = p^{\nu-1}.$$

From this we deduce the estimate

$$s_{k+1,\nu-1}(G)\geq s_{k,\nu}(G)p^{h-1},\quad \nu\geq 1, k\geq 0. \tag{2.4}$$

Next, let U be a subgroup counted by $s_{k,0}(G)$. These subgroups are in 1–1 correspondence with subgroups of $G/\langle x_h\rangle$ of index p^{k-a_U}, where $a_U=(\langle x_h\rangle : U\cap\langle x_h\rangle)$. Fix a_U, and consider all subgroups of $G/\langle x_h\rangle$ of index p^{k+1-a_U}. By our inductive hypothesis, the number of these subgroups is larger by a factor $\beta(p,h-1)$ than the number of subgroups of index p^{k-a_U}, and, changing, if necessary, the power of x_h occurring, we find that all these subgroups correspond to subgroups of index p^{k+1} in G. Hence,

$$s_{p^{k+1}}(G)\geq s_{k,0}(G)\beta(p,h-1),\quad k\geq 0. \tag{2.5}$$

Summing (2.4) over $\nu\geq 1$, and adding (2.5), we obtain

$$s_{p^k}(G)\leq\left(\beta(p,h-1)^{-1}+p^{1-h}\right)s_{p^{k+1}}(G),$$

and therefore

$$\beta(p,h)\geq\left(\beta(p,h-1)^{-1}+p^{1-h}\right)^{-1}.$$

Define $\beta^*(p,h)$ by $\beta^*(p,3)=p$ and

$$\beta^*(p,h)=\left(\beta^*(p,h-1)^{-1}+p^{1-h}\right)^{-1},\quad h\geq 4.$$

Then $\beta(p,h) \geq \beta^*(p,h)$, and, for $p \to \infty$, we have

$$\beta^*(p,h) \geq p \prod_{h=4}^{\infty} \left(1 + p^{2-h}\right)^{-1} = p + \mathcal{O}(1).$$

Finally, we have $\beta^*(p,h) \geq \beta^*(2,h)$, and by induction on h we obtain

$$\beta^*(p,h) \geq \beta^*(2,h) = \frac{2^{h-1}}{3 \cdot 2^{h-3} - 1} \searrow \frac{4}{3}. \qquad \square$$

3 Estimates for homomorphism numbers of nilpotent groups

Our next result is a slightly improved version of the main result in [4]. We write $\log_k$ for the k-fold iterated logarithm, that is, $\log_1 x = \log x$, and $\log_{k+1} x = \log \log_k x$.

Proposition 3.1 *Let G be a finitely generated nilpotent group, and let α be the abscissa of convergence of $\zeta_G(s) = \sum s_n(G) n^{-s}$. Then there exists some $\delta > 0$, such that $\zeta_G(s)$ can be continued meromorphically to* $\operatorname{Re} s > \alpha - \delta$, *and such that α is the only pole in the domain*

$$\left\{ s + it : s > \alpha - \frac{\delta}{\log^{2/3}(3 + |t|) \log_2^{1/3}(3 + |t|)} \right\}.$$

In particular, there exist polynomials P_1 and P_2, such that

$$\sum_{n \leq x} s_n(G) = P_1(\log x) x^\alpha + \mathcal{O}\left(x^\alpha e^{-c \log^{3/5} n \log_2^{-1/3} n}\right)$$

and

$$\sum_{n \geq 1} s_n(G) e^{-n/x} = P_1(\log x) x^\alpha + \mathcal{O}\left(x^\alpha e^{-c \frac{\log n}{\log_2^{2/3} n \log_3^{1/3} n}}\right).$$

Proof: It is shown in the proof of [4, Theorem 4.16], that there exist Hecke characters $\chi_1, \dots, \chi_r, \psi_1, \dots, \psi_s$, constants $a_1, \dots, a_r, b_1, \dots, b_r, c_1, \dots, c_s, d_1, \dots, d_s$ and a Dirichlet series $D(s)$, such that $D(s)$ converges absolutely for $\operatorname{Re} s \geq \alpha - \delta$, and such that

$$\zeta_G(s) = \frac{L(a_1 s + b_1, \chi_1) \cdots L(a_r s + b_r, \chi_r)}{L(c_1 s + d_1, \psi_1) \cdots L(c_r s + d_r, \psi_r)} D(s).$$

For Hecke L-series, Hinz [6] proved a zero-free region of Vinogradoff–Korobov-type,[2] that is, for a Hecke-character χ there exists a constant $c > 0$, such that $L(\sigma + it, \chi) \neq 0$

[2] Confer [2] for a more general result.

for $\sigma > 1 - \frac{c}{\log^{2/3}(|t|+3)\log_2^{1/3}(|t|+3)}$. Note that, since we consider a fixed set of characters, potential Siegel zeros can only alter the value of c. Hence, $\zeta_G(s)$ is holomorphic in this region with the exception of a pole at α. Moreover, from the functional equation for Hecke L-series we find that $|L(\sigma+it,\chi)| \ll 1+|t|^c$ uniformly in $\mathrm{Re}\, s > 0$. Define the path

$$\gamma = \left\{\alpha - \frac{\delta}{\log^{2/3}(3+|t|)\log_2^{1/3}(3+|t|)}, -\infty < t < \infty\right\}.$$

Then we deduce that for δ sufficiently small, $\zeta_G(s)$ is holomorphic to the right of γ with the exception of a pole at α, and that on γ we have the estimate $|\zeta_G(s)| \ll 1+|t|^c$. From Perron's formula[3] we find that

$$\begin{aligned}\sum_{n\le x} s_n(G) &= \int\limits_{\alpha+\frac{1}{\log x}-iT}^{\alpha+\frac{1}{\log x}+iT} \zeta_G(s)\frac{x^s}{s}\,ds + \mathcal{O}\left(\frac{x^\alpha \log x}{T}\right)\\ &= \operatorname{res}\, \zeta_G(s)\frac{x^s}{s}\Big|_{s=\alpha} + \int\limits_{\gamma\cap\{|\mathrm{Im}s|<T\}} \zeta_G(s)\frac{x^s}{s}\,ds + \mathcal{O}\left(\frac{x^\alpha \log x}{T}\right)\\ &= P_1(\log x)x^\alpha + \mathcal{O}\left(T^c x^\alpha e^{-c\frac{\log x}{\log^{2/3} T \log_2^{1/3} T}}\right) + \mathcal{O}\left(\frac{x^\alpha \log x}{T}\right),\end{aligned}$$

and choosing $T = \exp(c\log^{3/5} x \log_2^{-1/3} x)$ we obtain our claim. The proof of the second estimate is similar, now using the Mellin-transform

$$e^{-\tau} = \frac{1}{2\pi i}\int\limits_{1-i\infty}^{1+i\infty} \tau^{-s}\Gamma(s)\,ds$$

instead of Perron's formula. □

The following simple Tauberian theorem allows us to compare the coefficients of power series diverging at similar speed.

Lemma 3.2 *Let $f = \sum a_n z^n$, $g = \sum b_n z^n$, and $h = \sum c_n z^n$ be non-vanishing analytic functions in the unit circle, and suppose that the sequences (a_n), (b_n) are positive and non-decreasing, and that h is non-decreasing in $(0,1)$. Suppose that, for $z \nearrow 1$ we have*

$$\log f(z) = \log g(z) + \mathcal{O}(\log h(z)).$$

Then we have

$$\log a_n = \log b_n + \mathcal{O}(\log n + \log d_n),$$

where

$$d_n = \min_{0<z<1} z^{-n} h(z).$$

[3] Cf., for instance, [9, Lemma 3.12].

Proof: Define $h^*(z) = \frac{f(z)}{g(z)} = \sum c_n^* z^n$. Then $h^*(z) \leq h(z)^C$ for some constant C and $z \in (0, 1)$, and we obtain

$$c_n^* \leq \min_{0<z<1} z^{-n} h(z)^C \leq d_n^C.$$

Hence, a_n is the Cauchy convolution of b_n with a sequence bounded above by d_n^C, and we obtain

$$a_n = \sum_{\nu=0}^{n} b_\nu c_{n-\nu}^* \leq n b_n d_n^C,$$

and therefore

$$\log a_n \leq \log b_n + \mathcal{O}(\log n + \log d_n).$$

By symmetry, our claim follows. □

Theorem 3.3 *Let G be a finitely generated torsion-free nilpotent group. Then there exists a rational number α and a polynomial P, such that*

$$\log(h_n(G)) = \left(1 + \mathcal{O}\left(e^{-c\frac{\log n}{\log_2^{2/3} n \log_3^{1/3} n}}\right)\right) P(\log y)\, y^{1-\alpha}$$

where y is the unique solution in the interval $0 < y < 1$ *of the equation*

$$2\pi n = y^{-\alpha}\left((\alpha - 1)P(\log y) - P'(\log y)\right). \tag{3.1}$$

In particular, we have

$$\log h_n(G) = \left(C + \mathcal{O}\left(\frac{1}{\log n}\right)\right) n^{(\alpha-1)/\alpha} \log^{d/\alpha} n$$

for some positive constant C.

Proof: From Lemma 2.1 we find that $h_n(G)$ equals the coefficient of z^n in the product (2.1). We have

$$H(z) = \exp\left(\sum_{n=1}^{\infty}\sum_{\nu=1}^{\infty} \eta(n)\frac{z^{n\nu}}{\nu}\right),$$

where the series converges for $|z| < 1$, and all coefficients are real and positive. Setting $f(\tau) = H(e^{-\tau})$, we have

$$\begin{aligned} \log f(\tau) &= \sum_{n=1}^{\infty}\sum_{\nu=1}^{\infty} \eta(n)\frac{e^{-\tau n\nu}}{\nu} \\ &= \frac{1}{2\pi i}\int_{1+\alpha-i\infty}^{1+\alpha+i\infty} \tau^{-s}\,\Gamma(s)\,\zeta_G(s+1)\,ds, \end{aligned}$$

where we have used the Mellin-transform

$$e^{-\tau} = \frac{1}{2\pi i} \int\limits_{1-i\infty}^{1+i\infty} \tau^{-s}\Gamma(s)\, ds,$$

the interchanging of sum and integral being justified by absolute convergence. Let γ be the path

$$\gamma = \left\{\alpha - 1 - \frac{\delta}{\log^{2/3}(|t|+3)\log_2^{1/3}(|t|+3)}, -\infty < t < \infty\right\},$$

where δ is chosen small enough to ensure that $\zeta_G(s)$ has no singularities to the right of γ with the exception of $\alpha - 1$. Shifting the line of integration to γ we find that for $|\tau| \to 0$

$$\begin{aligned}\log f(\tau) &= \operatorname{res} \tau^{-s}\,\Gamma(s)\,\zeta_G(s+1)\Big|_{s=\alpha-1} + \frac{1}{2\pi i}\int\limits_{\gamma} \tau^{-s}\,\Gamma(s)\,\zeta_G(s+1)\,ds \\ &= P(\log\tau)\,\tau^{1-\alpha} + \mathcal{O}\left(\tau^{1-\alpha} e^{-\frac{c|\log\tau|}{\log^{2/3}|\log\tau|\log_2^{1/3}|\log\tau|}}\right),\end{aligned}$$

where P is a polynomial with positive leading coefficient. Reintroducing z, we obtain

$$\log H(z) = P(\log(1-z))(1-z)^{1-\alpha} + \mathcal{O}\left((1-z)^{1-\alpha} e^{-\frac{c|\log(1-z)|}{\log|\log(1-z)|}}\right).$$

Since $\eta(1) = 1$, the sequence $h_n(G)$ is non-decreasing, and we can apply Lemma 3.2. We have

$$\begin{aligned}&\min_{0<z<1} z^{-n} \exp\left((1-z)^{1-\alpha} e^{-\frac{c|\log(1-z)|}{\log^{2/3}|\log(1-z)|\log_2^{1/3}|\log(1-z)|}}\right) \\ &\qquad = z_0^{-n} \exp\left((1-z_0)^{1-\alpha} e^{-\frac{c|\log(1-z_0)|}{\log^{2/3}|\log(1-z_0)|\log_2^{1/3}|\log(1-z_0)|}}\right),\end{aligned}$$

where z_0 is the minimum of the function $z^{-n}\exp\big((1-z)^{1-\alpha}\big)$, which we approximate by $1 - \left(\frac{\alpha-1}{n}\right)^{1/\alpha}$. We deduce

$$\log h_n(G) = \log\Big([z^n]\exp\Big(P(\log(1-z))(1-z)^{1-\alpha}\Big)\Big) + \mathcal{O}\left(n^{\frac{\alpha-1}{\alpha}} e^{-c\frac{\log n}{\log_2^{2/3}\log_3^{1/3} n}}\right).$$

Let D be a Dirichlet series, holomorphic in the complex plane with a single pole at α, such that D and ζ_G have the same principal part in α. For instance, we may choose D to be a polynomial in $\zeta(s-\alpha+1)$. Then, we can repeat our argument with ζ_G replaced by D, hence, it suffices to prove our assertion under the assumption that ζ_G has holomorphic continuation to the entire plane. Arguing as above, but shifting the line of integration to $-1+\epsilon+it$, we obtain

$$\log f(\tau) = P(\log\tau)\tau^{1-\alpha} + \mathcal{O}\left(\tau^{1-\epsilon}\right).$$

Set $\tau = y+ix$. We show that $f(\tau)$ is small compared to $f(y)$, unless x is very small. Suppose first that $x > y^{4/3}$. From Lemma 2.2 and the definition of $\eta(n)$ we find that $\eta(n) > 1$ for almost all n, and therefore

$$\begin{aligned}\mathrm{Re}\sum_{n=1}^{\infty}\sum_{\nu=1}^{\infty}\eta(n)\frac{e^{-n\nu y+in\nu x}}{\nu} &\leq \mathrm{Re}\sum_{n=1}^{\infty}\sum_{\nu=1}^{\infty}\eta(n)\frac{e^{n\nu y}}{\nu} - \sum_{n:\eta(n)\geq 1} e^{-ny}(1-\mathrm{Re}\, e^{inx})\\ &\leq \log f(y) - \frac{cx^2}{y^3},\end{aligned}$$

that is, $|f(\tau)| \leq f(y)e^{-cy^{-1/3}}$. For smaller values of x, we use the Taylor series for $\log z$ to obtain

$$\begin{aligned}&\log f(\tau) = \log f(y) + ixy^{-\alpha}\left((1-\alpha)P(\log y) + P'(\log y)\right)\\ &+ x^2y^{-\alpha-1}\left(\alpha(1-\alpha)P(\log y) + (2\alpha-1)P'(\log y) - P''(\log y)\right) + \mathcal{O}\left(x^3y^{-\alpha-2}\right).\end{aligned} \tag{3.2}$$

We have

$$h_n(G) = \int_{-1/2}^{1/2} f(y+2\pi ix)e^{ny+2\pi inx}\,dx, \tag{3.3}$$

and we choose y such that the factor $e^{2\pi inx}$ and the linear term in (3.2) cancel, that is, we fix y as in (3.1). From what we have shown so far we know that we may restrict the range of integration in (3.3) to the interval $(-y^{2/3}, y^{2/3})$, and in this range we can apply (3.2) to obtain

$$h_n(G) = \left(1+\mathcal{O}\left(y^{1/3}\right)\right) f(y)\int_{-y^{2/3}}^{y^{2/3}} \exp\left(x^2y^{-\alpha-1}Q(\log y)\right)\,dx,$$

where

$$Q(\log y) = \left(\alpha(1-\alpha)P(\log y) + (2\alpha-1)P'(\log y) - P''(\log y)\right)$$

is negative for y sufficiently small. We deduce that

$$\log h_n(G) = \log f(y) + \mathcal{O}(\log n),$$

which is far better than needed. Finally, if d is the degree of P, we have $y \asymp \left(\frac{n}{\log^d n}\right)^{-1/\alpha}$, and therefore

$$\log f(y) \sim cy^{1-\alpha}\log^d y \asymp n^{(\alpha-1)/\alpha},$$

that is,

$$\log f(y) + \mathcal{O}\left(n^{(\alpha-1)/\alpha} e^{-c\frac{\log n}{\log\log n}}\right) = \left(1 + \mathcal{O}\left(e^{-c\frac{\log n}{\log_2^{2/3} n \log_3^{1/3} n}}\right)\right) f(y),$$

which proves our claim. □

For free abelian groups we can do better.

Proposition 3.4 *For $k \geq 2$, there exists an asymptotic expansion*

$$\log h_n\left(\mathbb{Z}^k\right) \approx \Theta(k)\left(1 + \sum_{\nu=1}^{\infty} \beta_\nu y^{k\nu}\right),$$

where

$$\Theta(k) = \sqrt{\frac{\pi}{A}}\, e^{r_1 y^{-k} + r_2 y^{-k+1} + ny}\, y^{\frac{k+2}{2}},$$

y is the unique positive solution of the equation

$$k r_1 y^{-k-1} + (k-1) r_2 y^{-k} = n, \tag{3.4}$$

and

$$\begin{aligned} r_1 &= (k+1)!\,\zeta(2)\cdots\zeta(k-1), \\ r_2 &= -\frac{1}{2}\,k!\,\zeta(2)\cdots\zeta(k-2), \\ A &= r_1\binom{k+1}{2} - r_2 y\binom{k}{2}. \end{aligned} \tag{3.5}$$

Proof: We have[4]

$$\zeta_{\mathbb{Z}^k}(s) = \zeta(s)\,\zeta(s-1)\cdots\zeta(s-k+1),$$

and since $\zeta(s) = 0$ for all negative even integers, we see that $\zeta_{\mathbb{Z}^k}(s)$ has simple poles at $s = k$ and $s = k-1$ with residuum $r_1 = \zeta(2)\cdots\zeta(k)$ and

$$r_2 = \zeta(0)\,\zeta(2)\,\zeta(3)\cdots\zeta(k-1) = -\frac{1}{2}\,\zeta(2)\,\zeta(3)\cdots\zeta(k-1),$$

respectively, and that $\zeta_{\mathbb{Z}^k}(s) = 0$ for all integers $n \leq \max(0, k-4)$. Hence, $\zeta_{\mathbb{Z}^k}(s)\Gamma(s+1)$ is holomorphic in the entire plane with the exception of the two poles aforementioned. Fix an integer $\ell > 1$. From the functional equation of ζ we deduce that for $\mathrm{Re}\, s = -\frac{1}{2} - \ell$, we have for $\arg\tau < \frac{\pi}{4}$ the bound

$$|\Gamma(s+1)\zeta_{\mathbb{Z}^k}(s)| \ll |s|^{k^2}\,\Gamma(-s/2)^k \ll_\ell (1+|t|)^{k^2} e^{-\frac{\pi}{4}kt},$$

[4] Cf. [5, Proposition 1.1] or [7, Chapter 15].

and we can estimate the integral

$$\int_{-\frac{1}{2}-\ell-i\infty}^{-\frac{1}{2}-\ell+i\infty} \tau^{-s}\Gamma(s+1)\zeta_{\mathbb{Z}^k}(s+1)\,ds \ll_\ell |\tau|^{\frac{1}{2}+\ell} \int_{-\infty}^{\infty} (1+|t|)^{k^2} e^{-\frac{\pi}{4}kt}\,dt \ll |\tau|^{1/2+\ell}.$$

Hence, we deduce that

$$f(\tau) = r_1\tau^{-k} + r_2\tau^{-k+1} + \mathcal{O}\left(|\tau|^{1/2+\ell}\right),$$

where r_1 and r_2 are given in (3.5). If $x > y^{4/3}$, we have $|f(\tau)| \leq f(y)e^{-y^{-1/3}}$ as in the proof of Theorem 3.3. After these preparations the proof of Proposition 3.4 mimics the proof of Meinardus' Theorem as given in [1, Chapter 6]. We have

$$h_n\left(\mathbb{Z}^k\right) = \int_{-1/2}^{1/2} f(y+2\pi ix)e^{ny+2\pi inx}dx,$$

and use our estimates for $f(\tau)$ first to bound the integral over the range $|x| > y^{4/3}$, and then to transform the integrand. Defining y as in (3.4) and substituting $\omega = xy^{-\frac{k+2}{2}}$, we finally obtain

$$h_n\left(\mathbb{Z}^k\right) = \left(1+\mathcal{O}_\ell\left(|\tau|^{1/2+\ell}\right)\right) f(y)e^{ny}y^{\frac{k+2}{2}} \int_{-y^{-\frac{k}{2}+\frac{1}{3}}}^{y^{-\frac{k}{2}+\frac{1}{3}}} \exp\left(-A\omega^2 - \Phi(\omega)\right)\,d\omega$$

where

$$\Phi(\omega) = r_1\sum_{\nu=3}^{\infty}\binom{-k}{\nu}\omega^2\left(i\omega y^{k/2}\right)^{\nu-2} + r_2y\sum_{\nu=3}^{\infty}\binom{-k+1}{\nu}\omega^2\left(i\omega y^{k/2}\right)^{\nu-2}$$

and

$$A = r_1\binom{-k}{2} - r_2y\binom{-k+1}{2}.$$

Set

$$e^{-\Phi(z)} = \sum_{\nu=0}^{\infty}\alpha_\nu z^\nu.$$

Then we have $\alpha_1 = \alpha_2 = 0$, and

$$\alpha_\nu \ll y^{\frac{k}{2}\left(\left[\frac{\nu}{3}\right]+(\nu \bmod 3)\right)},$$

and truncating this series after $\nu = N$, we introduce an error of size $y^{(N-1)k/6}$. Hence, we obtain

$$\int_{-y^{-\frac{k}{2}+\frac{1}{3}}}^{y^{-\frac{k}{2}+\frac{1}{3}}} \exp\left(-A\omega^2 - \Phi(\omega)\right)\, d\omega = \sum_{\nu=0}^{N} \alpha_\nu \int_{-y^{-\frac{k}{2}+\frac{1}{3}}}^{y^{-\frac{k}{2}+\frac{1}{3}}} e^{-A\omega^2}\omega^\nu\, d\omega + \mathcal{O}\left(y^{(N-1)k/6}\right).$$

By symmetry, the integral vanishes for odd ν, and we deduce the existence of an asymptotic expansion

$$\int_{-y^{-\frac{k}{2}+\frac{1}{3}}}^{y^{-\frac{k}{2}+\frac{1}{3}}} \exp\left(-A\omega^2 - \Phi(\omega)\right)\, d\omega \approx \sqrt{\frac{\pi}{A}}\left(1 + \sum_{\nu=1}^{\infty} \beta_\nu y^{k\nu}\right).$$

Since ℓ can be chosen arbitrarily large, our claim follows. □

For example, for $k = 3$, setting $x = y^{-1}$ we have to solve

$$96\zeta(2)\zeta(3)x^4 - 3\zeta(2)x^3 = n,$$

which can be expanded into an asymptotic series in $n^{-1/4}$ as

$$\begin{aligned} x = \frac{1}{\zeta(3)}\Bigg(&\frac{1}{2}\left(\frac{\zeta(3)^{3/4}n^{1/4}}{\pi^{1/2}}\right) + \frac{1}{128} + \frac{3}{16384}\left(\frac{\zeta(3)^{3/4}n^{1/4}}{\pi^{1/2}}\right)^{-1} \\ &+ \frac{1}{262144}\left(\frac{\zeta(3)^{3/4}n^{1/4}}{\pi^{1/2}}\right)^{-2} + \frac{15}{268435456}\left(\frac{\zeta(3)^{3/4}n^{1/4}}{\pi^{1/2}}\right)^{-3} \\ &- \frac{77}{2199023255552}\left(\frac{\zeta(3)^{3/4}n^{1/4}}{\pi^{1/2}}\right)^{-5} - \frac{3}{2199023255552}\left(\frac{\zeta(3)^{3/4}n^{1/4}}{\pi^{1/2}}\right)^{-6} \\ &- \frac{1989}{72057594037927936}\left(\frac{\zeta(3)^{3/4}n^{1/4}}{\pi^{1/2}}\right)^{-7}\Bigg) + \mathcal{O}\left(n^{-9/4}\right). \end{aligned}$$

Next, we compute α_ν for even values ν. We have $\alpha_2 = 0$, and

$$\begin{aligned} \alpha_4 &= \frac{5}{2}(96\zeta(3) - 1)\pi^2 y^3 \\ \alpha_6 &= -\frac{1}{2}\pi^2 y^3\left(896\zeta(3)y^3 - 7y^3 + 25600\pi^2(\zeta(3))^2 - 640\pi^2\zeta(3) + 4\pi^2\right), \end{aligned}$$

and $\alpha_\nu \ll y^6$ for $\nu \geq 8$. From this we obtain

$$\int_{-y^{-\frac{k}{2}+\frac{1}{3}}}^{y^{-\frac{k}{2}+\frac{1}{3}}} \exp\left(-A\omega^2 - \Phi(\omega)\right)\, d\omega = \sqrt{\frac{\pi}{A}}\left(1 + \frac{3}{4}\alpha_4 A^{-2} + \frac{15}{8}\alpha_6 A^{-3}\right) + \mathcal{O}\left(y^6\right),$$

that is,

$$\beta_1 = \frac{81920\zeta(3)^2 - 2240\zeta(3) + 14}{27\pi^2(64\zeta(3)-1)^3}.$$

Putting our estimate for y and β_1 into Proposition 3.4, we obtain

$$\begin{aligned} h_n\left(\mathbb{Z}^3\right) = {} & \frac{\sqrt{2}}{12\pi^{9/4}\zeta(3)^{5/8}n^{5/8}(64\zeta(3)-1)} \\ & \times \exp\left(4\sqrt{\pi}\sqrt[4]{\zeta(3)}n^{3/4} - \frac{\pi\sqrt{n}}{16\sqrt{\zeta}(3)} - \frac{\pi^{3/2}\sqrt[4]{n}}{2048\zeta(3)^{5/4}}\right) \\ & \times \left(1 + \gamma_1 n^{-1/4} + \gamma_2 n^{-1/2} + \gamma_3 n^{-3/4} + \mathcal{O}\left(n^{-1}\right)\right) \end{aligned}$$

with

$$\begin{aligned} \gamma_1 &= \frac{\sqrt{2}\left(7\pi^2 + 1310720\zeta(3)^2\right)}{268435456\pi^{3/4}\zeta(3)^{27/8}} \\ \gamma_2 &= \frac{\left(35127296\pi^2\zeta(3)^2 + 49\pi^4 + 3092376453120\zeta(3)^4\right)}{18014398509481984\pi^{(}3/4)\zeta(3)^{49/8}} \\ \gamma_3 &= \frac{\sqrt{2}P\left(\zeta(3)^{1/4}, \pi^{1/2}\right)}{16320498564797493858533376\zeta(3)^{33/4}\pi^{21/8}\zeta(3)^{5/8}(64\zeta(3)-1)^3}, \end{aligned}$$

where

$$\begin{aligned} P(x,y) = {} & 618970019642690137449562112 0x^{41} - 1692496147460480844588646 40x^{37} \\ & + 2231318163155907363471360 0x^{36}y^7 + 1057810092162800527867904x^{33} \\ & - 1045930388979331576627200x^{32}y^7 + 3847335089670067322 88x^{28}y^{11} \\ & + 1634266232780205588480 0x^{28}y^7 - 180343832328284405 76x^{24}y^{11} \\ & - 8511803295730237440 0x^{24}y^7 + 1285810129207296x^{20}y^{15} \\ & + 281787238012944384x^{20}y^{11} - 60272349806592x^{16}y^{15} \\ & - 1467641864650752x^{16}y^{11} + 809238528x^{12}y^{19} + 941755465728x^{12}y^{15} \\ & - 4904976384x^8y^{15} - 37933056x^8y^{19} + 592704x^4y^{19} - 3087y^{19}. \end{aligned}$$

It can be seen from the computations leading to this example, that the main source of error is the approximation of the solutions of (3.4), whereas the contribution of the series $\sum \beta_\nu y^{k\nu}$ is comparably small and simple.

4 Subgroup growth of free products of nilpotent groups

Theorem 4.1 *Let $G_1, \ldots, G_r$ be finitely generated torsion-free nilpotent groups, and set $\Gamma = G_1 * \cdots * G_r$. Then there exist polynomials P_ν with positive leading coefficients, such that*

$$\log s_n(\Gamma) = (r-1)n\log n + \left(1 + \mathcal{O}\left(e^{-c\frac{\log n}{\log_2^{2/3} n \log_3^{1/3} n}}\right)\right)\sum_{\nu=1}^{r} P_\nu(\log y_\nu)y_\nu^{1-\alpha_\nu},$$

where y_ν is the unique solution of the equation

$$n = y_\nu^{-\alpha_\nu}\left((\alpha_\nu - 1)P_\nu(\log y_\nu) - P_\nu'(\log y_\nu)\right),$$

and α_ν is the abscissa of convergence of $\zeta_{G_\nu}(s)$. The polynomials P_ν can be computed from the principal part of the Laurent-expansion of $\zeta_{G_\nu}(s)$ in $s = \alpha_\nu$, and the degree of P_ν equals the order of the pole of $\zeta_{G_\nu}(s)$ in $s = \alpha_\nu$ diminished by 1. Moreover, if each of the groups G_ν is free abelian, there exists an asymptotic expansion

$$s_n(\Gamma) = n \cdot n!^{r-1} \prod_{\nu=1}^{r} \Theta(k_\nu)\left(1 + \sum_{\nu=1}^{\infty} a_\nu n^{-\nu/m}\right),$$

where k_ν is the rank of G_ν, $\Theta(k_\nu)$ is given in Proposition 3.4, and m is the least common multiple of $k_1, \dots, k_r$.

Proof: From Theorem 3.3 and the universal property of free products we find that $nh_n(\Gamma)$ satisfies the estimate stated for $s_n(\Gamma)$. Since by [8, Proposition 4] the asymptotics $s_n(\Gamma) \sim nh_n(\Gamma)$ holds true for all finitely generated groups having a non-trivial free factor, our first claim follows. Similarly, for the case of free abelian factors it suffices to show that there exists an asymptotic expansion

$$s_n(\Gamma) = nh_n(\Gamma)\left(1 + \sum_{\nu=1}^{\infty} a_\nu n^{-\nu/m}\right).$$

Comparing coefficients in (2.3), we obtain the transformation formula

$$s_n(\Gamma) = nh_n(\Gamma) - \sum_{\nu=1}^{n-1} s_\nu(\Gamma)h_{n-\nu}(\Gamma),$$

and, together with the asymptotic expansions for $h_n(G_\nu)$ in Proposition 3.4, our claim follows. □

As an example, we consider $\Gamma = \mathbb{Z}^2 * \mathbb{Z}^3$. From the Hardy–Ramanujan–Rademacher theorem (cf., for instance, [1, Theorem 5.1]) we deduce

$$h_n\left(\mathbb{Z}^2\right) = p(n) = \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2n/3}}\left(1 - \frac{1}{48\sqrt{n}} - \frac{\sqrt{3}}{\pi\sqrt{2}\sqrt{n}} + \mathcal{O}\left(\frac{1}{n}\right)\right),$$

and the first terms of the asymptotic expansion of $h_n(\mathbb{Z}^3)$ have been computed in the last section. We obtain

$$\begin{aligned} s_n\left(\mathbb{Z}^2 * \mathbb{Z}^3\right) &= nh_n\left(\mathbb{Z}^2 * \mathbb{Z}^3\right)\left\{1 + \mathcal{O}\left(n^{-1}\right)\right\} \\ &= \frac{\sqrt{2}}{48\sqrt{3}\pi^{9/4}\zeta(3)^{5/8} n^{13/8}(64\zeta(3) - 1)} \\ &\quad \times \exp\left(4\sqrt{\pi}\sqrt[4]{\zeta(3)}n^{3/4} - \frac{\pi(16\sqrt{2\zeta(3)} - \sqrt{3})n^{1/2}}{16\sqrt{3\zeta(3)}} - \frac{\pi^{3/2}n^{1/4}}{2048\zeta(3)^{5/4}}\right) \\ &\quad \times \left\{1 + \gamma_1 n^{-1/4} + \gamma_2 n^{-1/2} + \gamma_3 n^{-3/4} + \mathcal{O}\left(n^{-1}\right)\right\} \end{aligned}$$

where

$$\gamma_1 = \frac{\sqrt{2}\left(7\pi^2 + 1310720\zeta(3)^2\right)}{268435456\pi^{3/4}\zeta(3)^{27/8}},$$

$$\gamma_2 = \frac{\sqrt{2}P_2\left(\zeta(3)^{1/2}, \pi\right)}{54043195528445952\pi^{9/4}\zeta(3)^{49/8}},$$

$$\gamma_3 = \frac{\sqrt{2}P_3\left(\zeta(3)^{1/4}\pi^{1/2}\right)}{1632049856479749385853337 6\pi^{13/8}\zeta(3)^{71/8}(64\zeta(3)-1)^3},$$

with polynomials P_2, P_3 given by

$$\begin{aligned} P_2(x, y) = & -140737488355328yx^{11} - 3377699720527872\sqrt{3}\sqrt{2}x^{11} \\ & + 105381888y^4x^4 + 147y^6 + 9277129359360y^2x^8, \end{aligned}$$

$$\begin{aligned} P_3(x, y) = & \, 618970019642690137449562112 0x^{41} - 16924961474604808445886464 0x^{37} \\ & + 2231318163155907363471360 0x^{36}y^7 + 105781009216280052786790 4x^{33} \\ & - 1045930388979331576627200x^{32}y^7 + 384733508967006732288x^{28}y^{11} \\ & + 1634266232780205588480 0x^{28}y^7 - 18034383232828440576x^{24}y^{11} \\ & - 851180329573023744 00x^{24}y^7 - 435213295061266502894223360x^{21}y^5 \\ & - 104451190814703960694613606 40\sqrt{6}x^{21}y^3 + 1285810129207296x^{20}y^{15} \\ & + 281787238012944384x^{20}y^{11} + 2040062320599686732316672 0x^{19}y^5 \\ & + 4896149569439248157560012 80\sqrt{6}x^{19}y^3 - 232428975328740350361 6x^{17}y^9 \\ & - 557829540788976840867 84\sqrt{6}x^{17}y^7 - 318759737593701051924480x^{17}y^5 \\ & - 765023370224882524618752 0\sqrt{6}x^{17}y^3 - 60272349806592x^{16}y^{15} \\ & - 1467641864650752x^{16}y^{11} + 108951082185347039232x^{15}y^9 \\ & + 2614825972448328941568\sqrt{6}x^{15}y^7 + 166020696663385964544 0x^{15}y^5 \\ & + 398449671992126314905 60\sqrt{6}x^{15}y^3 - 1702360659146047488x^{13}y^9 \\ & - 40856655819505139712\sqrt{6}x^{13}y^7 + 809238528x^{12}y^{19} + 941755465728x^{12}y^{15} \\ & + 8866461766385664x^{11}y^9 + 212795082393255936\sqrt{6}x^{11}y^7 \\ & - 37933056x^8y^{19} - 4904976384x^8y^{15} + 592704x^4y^{19} - 3087y^{19}. \end{aligned}$$

References

[1] G. E. Andrews, *The Theory of Partitions*, Cambridge University Press, 1998.

[2] M. D. Coleman, A zerofree region for the Hecke L-functions, *Mathematika* **37** (1990), 287–304.

[3] A. Dress and T. Müller, Decomposable functors and the exponential principle, *Adv. in Math.* **129** (1997), 188–221.

[4] M. du Sautoy, F. Grunewald, Analytic properties of zeta functions and subgroup growth, *Ann. Math.* **152** (2000), 793–833.

[5] F. Grunewald, D. Segal, G. Smith, Subgroups of finite index in nilpotent groups, *Invent. Math.* **93** (1988), 185–223.

[6] J. G. Hinz, Eine Erweiterung des nullstellenfreien Bereiches der Heckeschen Zetafunktion und Primideale in Idealklassen, *Acta Arith.* **38** (1980/81), 209–254.

[7] A. Lubotzky, D. Segal, *Subgroup Growth*, Birkhäuser, 2003.

[8] T. W. Müller, J.-C. Schlage-Puchta, Some probabilistic subgroups of a finitely generated group, submitted.

[9] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, 2nd edition (revised by D. R. Heath-Brown), Oxford University Press, 1986.

[10] K. Wohlfahrt, Über einen Satz von Dey und die Modulgruppe, *Arch. Math.* **29** (1977), 455–457.

Thomas W. Müller
School of Mathematical Sciences
Queen Mary & Westfield College
University of London
Mile End Road
London E1 4NS
United Kingdom
T.W.Muller@qmul.ac.uk

Jan-Christoph Schlage-Puchta
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg
Germany
jcp@math.uni-freiburg.de