Arch. Math. 84 (2005) 289–291
0003–889X/05/040289–03
DOI 10.1007/s00013-004-1201-1

Archiv der Mathematik

Normal growth of large groups, II

By

THOMAS W. MÜLLER and JAN-CHRISTOPH SCHLAGE-PUCHTA

Abstract. We show that a large class of groups has normal subgroup growth of type $n^{\log n}$, which is the same as the growth type of a free group of rank 2.

For a finitely generated group Γ, denote by $s_n^{\lhd}(\Gamma)$ the number of normal subgroups of index n, and consider the summatory function $S_n^{\lhd}(\Gamma) = \sum_{\nu \leqq n} s_\nu^{\lhd}(\Gamma)$. Lubotzky [2] proved that $S_n^{\lhd}(F_k)$ is of type $n^{\log n}$. Here, F_k denotes the free group of rank $k \geqq 2$, and a function $f(n)$ is called of type $n^{\log n}$, if there are positive constants c_1, c_2 such that

$$n^{c_1 \log n} \leqq f(n) \leqq n^{c_2 \log n}, \quad n \geqq n_0.$$

In a previous paper we considered the question as to what extent Lubotzky's estimate is characteristic for free groups, and stated a result ([4, Theorem 1]) to the effect that this type of behaviour pertains in fact to all finitely generated groups containing a finite index subgroup projecting onto a group G whose pro-p completion $\widehat{G}$ is a non-abelian free pro-p group for some prime p. While we still believe this statement to be true, its proof unfortunately turns out to be incorrect, as was kindly pointed out to us by Avinoam Mann. This does not affect Theorems 2 and 3 of [4]. We have not been able to remedy the proof of [4, Theorem 1]; instead, we have been led to the following slightly weaker result in the same direction.

Theorem 1. *Let Γ be a finitely generated group, possessing a normal finite index subgroup Δ which maps surjectively onto a group G such that the pro-p completion $\widehat{G}$ of G is a non-abelian free pro-p group for some prime p not dividing $(\Gamma : \Delta)$. Then $S_n^{\lhd}(\Gamma)$ is of type $n^{\log n}$.*

The proof makes use of an auxiliary result, which generalizes the technique of obtaining lower bounds by counting subspaces in elementary subgroups, employed for instance in [1].

Lemma 1. *Let G be a finite group and let k be a finite field such that* $\mathrm{char}(k) \nmid |G|$. *Let $\rho : G \to GL(V)$ be a representation of G over k of dimension n. Then there exists a*

Mathematics Subject Classification (2000): 20E07.

constant $c > 0$ depending only on G and k such that V contains at least e^{cn^2} ρ-invariant subspaces.

Proof. Let $k = \mathrm{GF}(q)$ and set $|G| = g$. Write

$$\rho = \sum_{i=1}^{N} m_i \rho_i, \tag{1}$$

where $\{\rho_1, \dots, \rho_N\}$ is a full set of inequivalent representations of G over k which are irreducible over k. Then

$$\max_i m_i \geqq \frac{n}{N \max_i \dim \rho_\mathrm{i}} = \hat{c} n, \tag{2}$$

where $\hat{c}$ depends only on G and k. Let

$$V = \bigoplus_{i=1}^{N} \bigoplus_{j=1}^{m_i} V_i^{(j)}$$

be a decomposition of V into ρ-invariant subspaces corresponding to Equation (1). Suppose without loss of generality that $m_1 = \max_i m_i$. Given an integer ℓ, $1 \leqq \ell \leqq m_1$, choose elements $x_i = (v_{ij})_{j=1}^{m_1}$ in $\bigoplus\limits_{j=1}^{m_1} V_1^{(j)}$, $1 \leqq i \leqq \ell$. This can be done in $q^{(\dim \rho_1)\ell \mathrm{m}_1}$ different ways. Two such sets $\{x_1, \dots, x_\ell\}$ and $\{y_1, \dots, y_\ell\}$ generate the same kG-submodule of V if and only if

$$\begin{pmatrix} y_1 \\ \vdots \\ y_\ell \end{pmatrix} = \mathbf{T} \begin{pmatrix} x_1 \\ \vdots \\ x_\ell \end{pmatrix}$$

for some invertible $(\ell \times \ell)$ matrix $\mathbf{T}$ over kG. Clearly, $\mathbf{T}$ can be chosen in at most $q^{g\ell^2}$ ways. Hence, given ℓ, the above construction provides at least $q^{(\dim \rho_1)\ell \mathrm{m}_1 - \mathrm{g}\ell^2}$ different kG-submodules of V. Choosing $\ell = \left[\frac{m_1 \dim \rho_1}{2g}\right]$, this yields

$$q^{\frac{m_1^2 (\dim \rho_1)^2}{4g} + \mathcal{O}(m_1)}$$

subspaces. In conjunction with (2), our claim follows with $c = \frac{\hat{c}^2}{4g}$. $\square$

Note that the condition $p \nmid |G|$ is necessary, since there exist finite p-groups with indecomposable representations over $\mathrm{GF}(p)$ of arbitrarily high dimension.

Proof of Theorem 1. Let Δ and p be as in the theorem. For a pro-p-group $\widehat{G}$, let $P_n(\widehat{G})$ be the lower central p-series of $\widehat{G}$. Define arithmetic functions $s(n, k)$ and $r(n, k)$ via

$$p^{s(n,k)} = (\widehat{F}_k : P_{n+1}(\widehat{F}_k)), \quad r(n, k) = \dim(\mathrm{P}_{\mathrm{n}+1}(\widehat{\mathrm{F}}_\mathrm{k}) / \mathrm{P}_{\mathrm{n}+2}(\widehat{\mathrm{F}}_\mathrm{k})),$$

where $\widehat{F}_k$ is the free pro-p group of rank k. It is known that, for fixed k as n tends to infinity,

$$r(n,k) \sim \frac{k^{n+2}}{(k-1)n}$$
$$s(n,k) \sim \frac{k^{n+2}}{(k-1)^2 n} \sim \frac{r(n,k)}{k-1};$$

cf., for instance, [3, Sec. 3.4]. Define characteristic subgroups Π_n of Δ recursively via $\Pi_0 = \Delta$ and $\Pi_{n+1} = \prod_n^p [\Pi_n, \Delta]$, $n \geqq 0$. Note that the closure of Π_n in $\widehat{\Delta}$ is precisely $P_n(\widehat{\Delta})$. Fix $k \geqq 2$ such that Δ projects onto $\widehat{F}_k$. Then Π_n/Π_{n+1} projects onto $P_n(\widehat{F}_k)/P_{n+1}(\widehat{F}_k)$, and therefore $\dim(\Pi_n/\Pi_{n+1}) \geqq r(n-1,k)$. Let U be a subspace of $V = \Pi_n/\Pi_{n+1}$. Then $U\Pi_{n+1}$ is normal in Δ, and it is normal in Γ if and only if U is invariant under the induced action of Γ/Δ on V. By assumption, $p \nmid |\Gamma/\Delta|$, and Lemma 1 implies the existence of $e^{cr(n-1,k)^2}$ invariant subspaces U. If Δ is d-generated, then the index of Π_n in Δ is at most the index of $P_n(\widehat{F}_d)$ in $\widehat{F}_d$; that is, there are $e^{cr(n-1,k)^2}$ normal subgroups in Γ of index at most $p^{s(n,d)}|\Gamma/\Delta|$. Since, by assumption, $k \geqq 2$, we have $r(n,k) \geqq s(n,d)^c$ for some positive constant c. Moreover, $|\Gamma : \Pi_n| \geqq |\Gamma : \Pi_{n+1}|^c$ for some positive constant c, and we conclude that $S_{k_\nu}^{\triangleleft}(\Gamma) > e^{c \log k_\nu{}^2}$ holds for a sequence (k_ν) with $k_\nu < k_{\nu+1} < k_\nu^C$ for some absolute constant C, which implies the claimed lower bound.

The upper bound $S_k^{\triangleleft}(\Gamma) < e^{c \log^2 k}$ holds for all finitely generated groups Γ and follows immediately from the upper bound for finitely generated free groups in [2].

References

[1] A. Lubotzky, Subgroup growth and congruence subgroups. Invent. Math. **119**, 267–295 (1995).
[2] A. Lubotzky, Enumerating boundedly generated finite groups. J. Algebra **238**, 194–199 (2001).
[3] A. Lubotzky and D. Segal, Subgroup Growth. Basel 2003.
[4] T. W. Müller and J.-C. Schlage-Puchta, Normal growth of large groups. Arch. Math. **81**, 609–613 (2003).

Received: 21 June 2004

Thomas W. Müller
School of Mathematical Sciences
Queen Mary, University of London
Mile End Road
E1 4NS London
United Kingdom
T.W.Muller@qmul.ac.uk

Jan-Christoph Schlage-Puchta
Mathematisches Institut
Albert-Ludwigs-Universität
Eckerstr. 1
D-79104 Freiburg
Germany
jcp@math.uni-freiburg.de