Optimal integer partitions

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Abstract

Let n be a positive integer and let $g_1, \ldots, g_n$ be real numbers. The following problem integer partition problem (IPP) is studied: Find a partition of the integer $n = \sum_{i=1}^{n} i \cdot \lambda_i$ such that $\sum_{i=1}^{n} g_i \cdot \lambda_i$ is maximal. An extended variant of the IPP is the problem EIPP, where, as a secondary condition, the number $\sum_{i=1}^{n} \lambda_i$ of items has to be minimal. The support of the partition is the index-set of all nonzero items, i.e. $\{i : \lambda_i > 0\}$. It is proved that there is always an optimal solution for the IPP (as well as for the EIPP) whose support contains at most $\lfloor \log_2(n+1) \rfloor$ elements and that this bound is sharp. An algorithm of time complexity $O(n^2)$ for the determination of such an optimal solution is presented. Finally the following non-polynomial bounds for the maximum number $M(n)$ of all optimal solutions for the EIPP are proved: $2.2324 n^{1/3} \lesssim \ln M(n) \lesssim \frac{1}{3}\sqrt[3]{6} n^{1/3} \ln n$ as $n \to \infty$.

Keywords: integer partition, optimal partition, restricted partition, dynamic programming, lexicographic order
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1. Introduction

Let P be a finite set of n objects and let $g_1, \dots, g_n$ be real numbers. Assume that every nonempty subset S of P generates the gain $g_{|S|}$, depending only on the size $|S|$ of S. Moreover, the gain of a partition π of P, i.e. of a disjoint union $P = P_1 \dot{\cup} \dots \dot{\cup} P_m$ of nonempty subsets of P is defined to be the sum of the gains of its classes, i.e.

$$G(\pi) = \sum_{k=1}^{m} g_{|P_k|}.$$

In this paper the following objective is studied: Determine a partition of P with a maximal gain which has in addition a small number of class-sizes. We may include the secondary condition, that the number of classes of the partition is minimal.

For a partition π of P let λ_i be the number of classes of size i. Note that $\sum_{i=1}^{n} i\lambda_i = n$ and that the coefficients λ_i are nonnegative integers, i.e. the partition of the set P generates a partition of the integer n into λ_i parts of size i, $i \in \{1, \dots, n\}$. We have $G(\pi) = \sum_{i=1}^{n} g_i\lambda_i$. The number of classes of π is given by $\sum_{i=1}^{n} \lambda_i$ and the number of different class-sizes by $|\{i : \lambda_i > 0\}|$.

Let $[n] = \{1, \dots, n\}$, $[n]^* = \{0, 1, \dots, n\}$ and $\mathbb{N} = \{0, 1, 2, \dots\}$. Formally, for a *gain vector* $(g_1, \dots, g_n)$, the *integer partition problem* (IPP) reads as follows:

$$\begin{aligned} \sum_{i=1}^{n} g_i\lambda_i &\to \max, \quad s.t. \\ \sum_{i=1}^{n} i\lambda_i &= n, \\ \lambda_i \in \mathbb{N}, &\quad \forall i \in [n]. \end{aligned}$$

Special attention is directed to optimal solutions for which $|\{i : \lambda_i > 0\}|$ is "small".

For the *extended integer partition problem* (EIPP) the secondary condition reads

$$\sum_{i=1}^{n} \lambda_i \to \min,$$

i.e. this objective function is mimimized over the set of all optimal solutions of the IPP.

For two vectors $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$, let $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{n} a_i b_i$ be their inner product and $\operatorname{supp}(\boldsymbol{a}) = \{i : a_i \neq 0\}$ be the *support of* $\boldsymbol{a}$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, $\boldsymbol{g} = (g_1, \ldots, g_n)$, $\overrightarrow{\boldsymbol{n}} = (1, \ldots, n)$ and $\mathbf{1} = (1, \ldots, 1)$. With this notation the IPP reads:

$$\max\{\boldsymbol{g} \cdot \boldsymbol{\lambda} : \overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{\lambda} = n, \boldsymbol{\lambda} \in \mathbb{N}^n\}. \tag{IPP}$$

Moreover, the EIPP reads

$$\min\{\mathbf{1} \cdot \boldsymbol{\lambda} : \lambda \text{ is optimal for the IPP}\}. \tag{EIPP}$$

The IPP is similar to the knapsack problem (cf. [1]), but the essential difference is that the condition in the IPP has the form of an equality and its RHS is equal to the dimension n. We prove that there is always an optimal solution $\boldsymbol{\lambda}$ of the IPP and of the EIPP with $|\operatorname{supp}(\boldsymbol{\lambda})| \leq \log_2(n+1)$ and that this bound cannot be improved. A dynamic programming approach leads to an algorithm of time complexity $O(n^2)$. It is interesting that the average size of the support (extended over all partitions of the integer n) is essentially larger, namely $\frac{\sqrt{6n}}{\pi}(1 + o(1))$ as $n \to \infty$. This was proved by Wilf in [3].

Moreover, we present an algorithm that enumerates all optimal solutions. A survey on algorithms that generate all partitions of an integer, also with restrictions, is contained in the paper of Zoghbi and Stojmenović [6]. Finally, we prove non-polynomial bounds for the number of all optimal solutions in the worst case. Related questions of superpolynomial rates of restricted integer partition functions are discussed by Canfield and Wilf in [4].

A special case has been studied by Došlić in [2]: He maximized the product of the parts of an integer partition into distinct parts. Turning to the logarithm this leads to the IPP, where $g_i = \ln i$ and with the additional condition that $\lambda_i \leq 1$ for all i.

A comprehensive presentation of the theory of integer partitions can be found in the book of Andrews [5].

2. Preliminary results

Lemma 1. *There is a gain vector* $\boldsymbol{g}$ *such that* $|\operatorname{supp}(\boldsymbol{\lambda})| = \lfloor \log_2(n+1) \rfloor$ *for all optimal solutions* $\boldsymbol{\lambda}$ *of the IPP.*

Proof. The cases $n = 1$ and $n = 2$ are trivial, so let $n \geq 3$. Let $k = \lfloor \log_2(n+1) \rfloor$. Then $k \geq 2$ and $2^k - 1 \leq n \leq 2^{k+1} - 2$. Let

$$g_i = \begin{cases} 10^\ell & \text{if } i = 2^\ell \text{ for some } \ell \in \{0, \dots, k-2\}, \\ 10^{k-1} & \text{if } i = n - 2^{k-1} + 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\lambda_i^* = \begin{cases} 1 & \text{if } i = 2^\ell \text{ for some } \ell \in \{0, \dots, k-2\}, \\ 1 & \text{if } i = n - 2^{k-1} + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the vector $\boldsymbol{g}$ is well-defined because of the inequality $2^{k-2} < n - 2^{k-1} + 1$. Obviously, $\boldsymbol{\lambda}^*$ is an admissible solution of the IPP with gain $\boldsymbol{g} \cdot \boldsymbol{\lambda}^* = \sum_{\ell=0}^{k-1} 10^\ell = \frac{10^k - 1}{9}$ and $|\operatorname{supp}(\boldsymbol{\lambda})| = k$. We show that every other admissible solution $\boldsymbol{\lambda}$ of the IPP has smaller gain (and hence $\boldsymbol{\lambda^*}$ is the only optimal solution). Assume the contrary. There there is some optimal solution $\boldsymbol{\lambda}$ different from $\boldsymbol{\lambda}^*$ such that $\boldsymbol{g} \cdot \boldsymbol{\lambda} \geq \boldsymbol{g} \cdot \boldsymbol{\lambda}^*$.

For $j = n - 2^{k-1} + 1$ we have $\lambda_j \leq 1$ because otherwise

$$\overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{\lambda} \geq j \cdot 2 = n + n - 2^k + 2 > n,$$

a contradiction the admissibility of $\boldsymbol{\lambda}$. If $\lambda_j = 0$ then

$$\begin{aligned} \boldsymbol{g} \cdot \boldsymbol{\lambda} = \sum_{i=1}^{n} g_i \lambda_i \leq \sum_{\ell=0}^{k-2} 10^\ell \lambda_{2^\ell} \leq \sum_{\ell=0}^{k-2} 10^\ell \frac{n}{2^\ell} \\ = n \sum_{\ell=0}^{k-2} 5^\ell < 2^{k+1} \frac{5^{k-1} - 1}{4} < 10^{k-1} < \frac{10^k - 1}{9} = \boldsymbol{g} \cdot \boldsymbol{\lambda}^*, \end{aligned}$$

a contradiction to the optimality of $\boldsymbol{\lambda}^*$.

Consequently, $\lambda_j = 1$, and besides $j > 2^{k-2}$. Then $\lambda_{2^{k-2}} \leq 1$ because otherwise

$$\overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{\lambda} \geq 2^{k-2} \cdot 2 + (n - 2^{k-1} + 1) > n,$$

a contradiction to the admissibility of $\boldsymbol{\lambda}$. If $\lambda_{2^\ell} \leq \lambda_{2^\ell}^* = 1$ for all $\ell \in \{0, \dots, k-2\}$, then one of these inequalities is strict in view of $\boldsymbol{\lambda} \neq \boldsymbol{\lambda^*}$ and hence $\boldsymbol{g} \cdot \boldsymbol{\lambda} < \boldsymbol{g} \cdot \boldsymbol{\lambda}^*$, a contradiction. Thus there is some $\ell \in \{0, \ldots, k-3\}$ such that $\lambda_{2^\ell} \geq 2$. We define $\boldsymbol{\lambda}'$ by

$$\lambda_i' = \begin{cases} \lambda_i - 2 & \text{if } i = 2^\ell, \\ \lambda_i + 1 & \text{if } i = 2^{\ell+1}, \\ \lambda_i & \text{otherwise.} \end{cases}$$

Then $\boldsymbol{\lambda}'$ is admissible and

$$\boldsymbol{g} \cdot \boldsymbol{\lambda}' = \boldsymbol{g} \cdot \boldsymbol{\lambda} - 2 \cdot 10^\ell + 10^{\ell+1} > \boldsymbol{g} \cdot \boldsymbol{\lambda},$$

a contradiction to the optimality of $\boldsymbol{\lambda}$. □

In the following we use a *lexicographic ordering*. For two vectors $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ let $\boldsymbol{a} \prec \boldsymbol{b}$ if $a_i < b_i$ for the largest index i with $a_i \neq b_i$. As usual, $\boldsymbol{a} \preceq \boldsymbol{b}$ if $\boldsymbol{a} \prec \boldsymbol{b}$ or $\boldsymbol{a} = \boldsymbol{b}$. Note that in this definition components with larger (and not with smaller) index have higher priority.

Lemma 2. *Let $\boldsymbol{\lambda}$ be a lexicographically largest optimal solution of the IPP. Then*

$$|\operatorname{supp}(\boldsymbol{\lambda})| \leq \lfloor \log_2(n+1) \rfloor.$$

Proof. Let $\boldsymbol{x} \neq \boldsymbol{0}$ be an integer vector such that $0 \leq |x_i| \leq \lambda_i$ for all i. First we show that $\overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{x} \neq 0$. Assume the contrary. Then $\boldsymbol{\lambda}' = \boldsymbol{\lambda} - \boldsymbol{x}$ and $\boldsymbol{\lambda}'' = \boldsymbol{\lambda} + \boldsymbol{x}$ are also admissible solutions of the IPP. The equalities $\boldsymbol{g} \cdot \boldsymbol{\lambda}' = \boldsymbol{g} \cdot \boldsymbol{\lambda} - \boldsymbol{g} \cdot \boldsymbol{x}$, $\boldsymbol{g} \cdot \boldsymbol{\lambda}'' = \boldsymbol{g} \cdot \boldsymbol{\lambda} + \boldsymbol{g} \cdot \boldsymbol{x}$ and the optimality of $\boldsymbol{\lambda}$ imply that $\boldsymbol{g} \cdot \boldsymbol{x} = 0$ and hence that $\boldsymbol{\lambda}'$ and $\boldsymbol{\lambda}''$ are also optimal. But $\boldsymbol{\lambda}'$ or $\boldsymbol{\lambda}''$ is lexicographically larger than $\boldsymbol{\lambda}$, a contradiction. From this observation it follows that all numbers $\overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{x}$ are pairwise different if $\boldsymbol{x}$ runs through the set of all integer vectors with $\boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{\lambda}$. Clearly, we have for this set of $\prod_{i=1}^n (\lambda_i + 1)$ vectors $0 \leq \overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{x} \leq \overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{\lambda} = n$. Consequently,

$$\prod_{i=1}^{n} (\lambda_i + 1) \leq n + 1$$

and further

$$\prod_{i \in \operatorname{supp}(\boldsymbol{\lambda})} (1 + 1) \leq n + 1$$

which finally leads to

$$2^{|\operatorname{supp}(\boldsymbol{\lambda})|} \leq n + 1 \text{ and } |\operatorname{supp}(\boldsymbol{\lambda})| \leq \lfloor \log_2(n+1) \rfloor.$$

$\square$

Remark 1. *Obviously, Lemmas 1 and 2 are in the same way true for the EIPP.*

3. Recursions

From Lemmas 1 and 2 it follows that there is always an optimal solution $\boldsymbol{\lambda}$ of the IPP (and of the EIPP) with support size not greater than $\lfloor \log_2(n+1) \rfloor$ and that this bound cannot be improved. In the next section, we present an algorithm that determines such an optimal solution. In the sense of dynamic programming we replace the restriction $\overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{\lambda} = n$ by the restriction $\overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{\lambda} = k$, where $k \in [n]^*$. In this section, the necessary recursion formulas are derived. Let

$$\begin{aligned} \Lambda_k &= \{\boldsymbol{\lambda} \in \mathbb{N}^n : \overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{\lambda} = k\}, \\ w_k &= \max\{\boldsymbol{g} \cdot \boldsymbol{\lambda} : \boldsymbol{\lambda} \in \Lambda_k\}, \quad S_k = \operatorname{argmax}\{\boldsymbol{g} \cdot \boldsymbol{\lambda} : \boldsymbol{\lambda} \in \Lambda_k\}, \\ \ell_k &= \min\{\mathbf{1} \cdot \boldsymbol{\lambda} : \boldsymbol{\lambda} \in S_k\}, \quad S_k^* = \operatorname{argmin}\{\mathbf{1} \cdot \boldsymbol{\lambda} : \boldsymbol{\lambda} \in S_k\}, \end{aligned}$$

i.e. S_k is the set of all optimal solutions of the modified IPP and S_k^* is the set of all optimal solutions of the modified EIPP. Note that $\Lambda_0 = S_0 = S_0^* = \{\mathbf{0}\}$ and $w_0 = \ell_0 = 0$. Let

$$m(\boldsymbol{\lambda}) = \begin{cases} \max\{i : \lambda_i > 0\} = \max(\operatorname{supp}(\boldsymbol{\lambda})) & \text{if } \boldsymbol{\lambda} \neq \mathbf{0}, \\ 0 & \text{if } \boldsymbol{\lambda} = \mathbf{0} \end{cases}$$

be the largest position of a nonzero element of $\boldsymbol{\lambda}$. Let $[-1]$ and $[0]$ be defined as the empty set and let $g_0 = -1$. Moreover, let for $k \in [n]^*$

$$\begin{aligned} a_k &= \max\{m(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \in S_k\}, \quad M_k = \{m(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \in S_k\}, \\ a_k^* &= \max\{m(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \in S_k^*\}, \quad M_k^* = \{m(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \in S_k^*\}, \\ E_k &= \{i \in [k-1] : w_i = g_i\} \cup \{k\}, \\ I_k &= \operatorname{argmax}\{g_i + w_{k-i} : i \in E_k\}, \\ I_k^* &= \operatorname{argmin}\{1 + \ell_{k-i} : i \in I_k\}. \end{aligned}$$

Note that $a_0 = a_0^* = 0$ and $M_0 = M_0^* = E_0 = \{0\}$. In the following we use the notation $\boldsymbol{e}_i$ for the unit vector of dimension n whose i-th entry equals 1.

Lemma 3. *Let* $\boldsymbol{\lambda} \in S_k$ *and* $\lambda_i > 0$. *Then* $\boldsymbol{\lambda} = \boldsymbol{e}_i + \boldsymbol{\lambda}'$ *with some* $\boldsymbol{\lambda}' \in S_{k-i}$, $\boldsymbol{g} \cdot \boldsymbol{\lambda} = w_k = g_i + w_{k-i}$ *and* $i \in I_k$.

Proof. The equality $\boldsymbol{g} \cdot \boldsymbol{\lambda} = w_k$ is obvious. Let $\boldsymbol{\lambda}' = \boldsymbol{\lambda} - \boldsymbol{e}_i$. Then $\boldsymbol{\lambda}' \in \Lambda_{k-i}$. We show that even $\boldsymbol{\lambda}' \in S_{k-i}$. Assume the contrary. Then there is some $\boldsymbol{\lambda}'' \in \Lambda_{k-i}$ with $\boldsymbol{g} \cdot \boldsymbol{\lambda}' < \boldsymbol{g} \cdot \boldsymbol{\lambda}''$. But the vector $\boldsymbol{e}_i + \boldsymbol{\lambda}''$ belongs to Λ_k and has gain

$$\boldsymbol{g} \cdot (\boldsymbol{e}_i + \boldsymbol{\lambda}'') = g_i + \boldsymbol{g}\boldsymbol{\lambda}'' > g_i + \boldsymbol{g}\boldsymbol{\lambda}' = \boldsymbol{g} \cdot \boldsymbol{\lambda},$$

a contradiction to $\boldsymbol{\lambda} \in S_k$. Therefore in fact $\boldsymbol{\lambda} = \boldsymbol{e}_i + \boldsymbol{\lambda}'$ with some $\boldsymbol{\lambda}' \in S_{k-i}$. Hence, it follows that

$$\boldsymbol{g} \cdot \boldsymbol{\lambda} = \boldsymbol{g} \cdot (\boldsymbol{e}_i + \boldsymbol{\lambda}') = g_i + w_{k-i}.$$

Assume that $i \notin I_k$. Then $i \notin E_k$ or there is some $j \in E_k$ with $g_j + w_{k-j} > g_i + w_{k-i}$. If $i \notin E_k$ then $g_i < w_i$ and hence there is some vector $\boldsymbol{\lambda}''' \in \Lambda_i$ with $g_i < \boldsymbol{g} \cdot \boldsymbol{\lambda}'''$. Let $\boldsymbol{\lambda}' \in S_{k-i}$. Then the vector $\boldsymbol{\lambda}''' + \boldsymbol{\lambda}'$ belongs to Λ_k and has gain

$$\boldsymbol{g} \cdot (\boldsymbol{\lambda}''' + \boldsymbol{\lambda}') > g_i + w_{k-i} = \boldsymbol{g} \cdot \boldsymbol{\lambda},$$

a contradiction to $\boldsymbol{\lambda} \in S_k$. If $g_j + w_{k-j} > g_i + w_{k-i}$ for some $j \in E_k$ then for any vector $\boldsymbol{\lambda}'' \in S_{k-j}$ the gain of the vector $\boldsymbol{e}_j + \boldsymbol{\lambda}''$ equals

$$\boldsymbol{g} \cdot (\boldsymbol{e}_j + \boldsymbol{\lambda}'') = g_j + w_{k-j} > g_i + w_{k-i} = \boldsymbol{g} \cdot \boldsymbol{\lambda},$$

a contradiction to $\boldsymbol{\lambda} \in S_k$. □

Theorem 1. *We have for* $k \in [n]$

$$E_k = \begin{cases} E_{k-1} \cup \{k\} & \text{if } g_{k-1} = w_{k-1}, \\ E_{k-1} \cup \{k\} \setminus \{k-1\} & \text{otherwise}, \end{cases} \tag{1}$$

$$w_k = \max\{g_i + w_{k-i} : i \in E_k\}, \tag{2}$$

$$a_k = \max\{\max\{i, a_{k-i}\} : i \in I_k\}, \tag{3}$$

$$M_k = \bigcup_{i \in I_k} \bigcup_{j \in M_{k-i}} \{\max\{i, j\}\}. \tag{4}$$

Proof. The recursion formula (1) is trivial.

Proof of (2): Let

$$v_k = \max\{g_i + w_{k-i} : i \in E_k\}.$$

We have to prove that $v_k = w_k$ and start with the proof of $v_k \leq w_k$. Let $i \in I_k$ and $\boldsymbol{\lambda}' \in S_{k-i}$. Then $\boldsymbol{\lambda} = \boldsymbol{e}_i + \boldsymbol{\lambda}' \in \Lambda_k$ and

$$v_k = g_i + w_{k-i} = \boldsymbol{g} \cdot \boldsymbol{e}_i + \boldsymbol{g} \cdot \boldsymbol{\lambda}' = \boldsymbol{g} \cdot \boldsymbol{\lambda} \leq w_k.$$

Now we prove $v_k \geq w_k$. Let $\boldsymbol{\lambda} \in S_k$ and let $i = m(\boldsymbol{\lambda})$. By Lemma 3, $i \in I_k \subseteq E_k$ and $w_k = g_i + w_{k-i} \leq v_k$.

Proof of (3): Let

$$b_k = \max\{\max\{i, a_{k-i}\}, i \in I_k\}.$$

We have to prove that $a_k = b_k$ and start with the proof of $a_k \leq b_k$. We choose $\boldsymbol{\lambda} \in S_k$ such that $a_k = m(\boldsymbol{\lambda})$ and write briefly $i = a_k$. By Lemma 3, $i \in I_k$ and thus

$$a_k \leq \max\{i, a_{k-i}\} \leq b_k.$$

Now we prove $a_k \geq b_k$. Let $b_k = \max\{j, a_{k-j}\}$ for some $j \in I_k$. There is some $\boldsymbol{\lambda}' \in S_{k-j}$ such that $m(\boldsymbol{\lambda}') = a_{k-j}$. Since $j \in I_k$ we have $w_k = g_j + w_{k-j}$ (because of (2)) and thereby the vector $\boldsymbol{\lambda}'' = \boldsymbol{e}_j + \boldsymbol{\lambda}'$ belongs to S_k. Hence,

$$a_k \geq m(\boldsymbol{\lambda}'') = \max\{j, a_{k-j}\} = b_k.$$

Proof of (4): First we show "$\subseteq$". Let $i \in M_k$, i.e. there is some $\boldsymbol{\lambda} \in S_k$ such that $m(\boldsymbol{\lambda}) = i$. By Lemma 3, $i \in I_k$ and $\boldsymbol{\lambda} = \boldsymbol{e}_i + \boldsymbol{\lambda}'$ with some $\boldsymbol{\lambda}' \in S_{k-i}$. Let $j = m(\boldsymbol{\lambda}')$, i.e. $j \in M_{k-i}$. Then $m(\boldsymbol{\lambda}) = \max\{i, j\}$ and thus i is contained in the RHS of (4).

Now we show "$\supseteq$". Let $\ell = \max\{i, j\}$ for some $i \in I_k$ and $j \in M_{k-i}$. Then there is some $\boldsymbol{\lambda}' \in S_{k-i}$ with $j = m(\boldsymbol{\lambda}')$, and the vector $\boldsymbol{\lambda} = \boldsymbol{e}_i + \boldsymbol{\lambda}'$ belongs to S_k (because of (2) and the fact $i \in I_k$). Since $m(\boldsymbol{\lambda}) = \max\{i, j\} = \ell$ it follows $\ell \in M_k$. $\square$

Theorem 2. *We have for $k \in [n]$*

$$\ell_k = \min\{1 + \ell_{k-i} : i \in I_k\}, \tag{5}$$

$$a_k^* = \max\{\max\{i, a_{k-i}^*\}, i \in I_k^*\}, \tag{6}$$

$$M_k^* = \bigcup_{i \in I_k^*} \bigcup_{j \in M_{k-i}^*} \{\max\{i, j\}\}. \tag{7}$$

The proof is analogous to the proof of Theorem 1.

For $u \in [n]^*$, let $\boldsymbol{\mu}_u$ and $\boldsymbol{\mu}_u^*$ be the lexicographically maximal elements of S_u and S_u^*, respectively.

Theorem 3. *We have for* $k \in [n]$

$$\boldsymbol{\mu}_k = \boldsymbol{e}_{a_k} + \boldsymbol{\mu}_{k-a_k}, \tag{8}$$
$$\boldsymbol{\mu}_k^* = \boldsymbol{e}_{a_k^*} + \boldsymbol{\mu}_{k-a_k^*}^*. \tag{9}$$

Proof. We prove only the first equality. The second can be proved analogously. Let briefly $i = a_k$. By definition, there is some $\boldsymbol{\lambda} \in S_k$ with $m(\boldsymbol{\lambda}) = i$. By Lemma 3, $\boldsymbol{\lambda} = \boldsymbol{e}_i + \boldsymbol{\lambda}'$ with some $\boldsymbol{\lambda}' \in S_{k-i}$. Since also $\boldsymbol{\mu}_{k-i} \in S_{k-i}$, we have $\boldsymbol{g} \cdot \boldsymbol{\lambda}' = \boldsymbol{g} \cdot \boldsymbol{\mu}_{k-i}$ and thereby $\boldsymbol{g} \cdot \boldsymbol{\lambda} = \boldsymbol{g} \cdot (\boldsymbol{e}_i + \boldsymbol{\mu}_{k-i})$, whence $\boldsymbol{e}_i + \boldsymbol{\mu}_{k-i} \in S_k$. Assume that there is some element $\hat{\boldsymbol{\lambda}}$ of S_k with $\hat{\boldsymbol{\lambda}} \succ \boldsymbol{e}_i + \boldsymbol{\mu}_{k-i}$. Since $\boldsymbol{\mu}_{k-i} \succeq \boldsymbol{\lambda}'$ it follows $\hat{\boldsymbol{\lambda}} \succ \boldsymbol{\lambda}$, in particular $m(\hat{\boldsymbol{\lambda}}) \geq m(\boldsymbol{\lambda})$. The definition of a_k implies $m(\hat{\boldsymbol{\lambda}}) \leq m(\boldsymbol{\lambda})$, and hence $m(\hat{\boldsymbol{\lambda}}) = m(\boldsymbol{\lambda}) = i$. By Lemma 3, the vector $\hat{\boldsymbol{\lambda}}' = \hat{\boldsymbol{\lambda}} - \boldsymbol{e}_i$ belongs to S_{k-i}. From $\hat{\boldsymbol{\lambda}} \succ \boldsymbol{e}_i + \boldsymbol{\mu}_{k-i}$ it follows that $\hat{\boldsymbol{\lambda}}' \succ \boldsymbol{\mu}_{k-i}$, a contradiction to the fact that $\boldsymbol{\mu}_{k-i}$ is the lexicographically largest element of S_{k-i}. □

4. Algorithmic solution

Theorems 1 - 3 form the basis and prove the correctness of the following algorithms. For the solution of the IPP Theorem 1 and (8) of Theorem 3 are used. The necessary extensions for the EIPP are presented on the RHS in parentheses and built on Theorem 2 and (9) of Theorem 3.

Algorithm 1 Determination of the optimal value of the objective function

Real numbers $g_1, \ldots, g_n$. $w_0 := 0, a_0 := 0, g_0 := -1$. ($\ell_0 := 0$.)
$E := \{0\}$. # $M_0 := \{0\}$. **for all** $k = 1, \ldots, n$ **do**
$E := E \cup \{k\}$.
if $g_{k-1} \neq w_{k-1}$ **then**
$E := E \setminus \{k-1\}$.
end if
$w_k := \max\{g_i + w_{k-i} : i \in E\}$.
$I := \{i \in E : w_k = g_i + w_{k-i}\}$.
($\ell_k := \min\{1 + \ell_{k-i} : i \in I\}$.)
($I := \{i \in I : \ell_k = 1 + \ell_{k-i}\}$.)
$a_k := \max\{\max\{i, a_{k-i}\} : i \in I\}$.
$M_k = \cup_{i \in I} \cup_{j \in M_{k-i}} \{\max\{i, j\}\}$.
end forw_n.

Algorithm 2 Determination of the optimal solution

Integers $a_1, \ldots, a_n$ from Algorithm 1. $\boldsymbol{\lambda} := \mathbf{0}$. $k := n$. **while** $k > 0$ **do**
$j := a_k$, $\lambda_j := \lambda_j + 1$, $k := k - j$.
end while$\lambda_1, \ldots, \lambda_n$.

Algorithm 2 provides the lexicographically maximal solution of the IPP (resp. EIPP) which has, by Lemma 2, support size not greater than $\lfloor \log_2(n+1) \rfloor$. Without the determination of the sets M_k, Algorithms 1 and 2 have obviously time complexity $O(n^2)$. In order to compute all optimal solutions, we have to include the determination of M_k and then the time complexity becomes $O(n^3)$. All optimal solutions can be obtained in increasing lexicographic order by a simple recursive method:

Algorithm 3 Determination of all optimal solutions

Sets $M_1, \ldots, M_n$ from Algorithm 1. $\boldsymbol{\lambda} := \mathbf{0}$. explore$(n, n)$.

The recursive procedure *explore* is given as follows:

Algorithm 4 explore(n, s)

```
for all i ∈ M_n (in increasing order) do
  if i ≤ s then
    λ_i := λ_i + 1.
    j := n − i.
    if j = 0 then
      write(λ).
    else
      explore(j, i).
    end if
    λ_i := λ_i − 1.
  end if
end for
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Here "write($\boldsymbol{\lambda}$)" means that one optimal solution is found and can be written into the list of all optimal solutions. The second parameter s can be interpreted as the last used summand. The condition $i \leq s$ ensures that the summands of the integer partition are given in decreasing order, and hence repetitions of solutions cannot appear.

The correctness follows from the definition of the sets M_k. The time complexity of Algorithm 3 equals the number of all optimal solutions. Hence it is worth to know how many optimal solutions may exist. In the next section we present non-polynomial lower bounds for the number of optimal solutions of the IPP and of the EIPP.

5. Bounds for the number of optimal solutions

Let $N(\boldsymbol{g})$ be the number of all optimal solutions of the IPP or of the EIPP, respectively, and let

$$M(n) = \max\{N(\boldsymbol{g}) : \boldsymbol{g} \in \mathbb{R}^n\}.$$

Note that $M(n)$ cannot be bounded by a polynomial in n iff the quotient $\ln M(n)/\ln n$ tends to infinity for $n \to \infty$.

For the IPP, the determination of $M(n)$ is trivial: Let $g_i = i$ for all i. Then the gain of each partition equals n and hence each partition is optimal.

Consequently, $M(n)$ is equal to the number $p(n)$ of partitions of an integer n which is given by the Hardy-Ramanujan formula [7]

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \text{ as } n \to \infty.$$

It follows that $\ln M(n)$ is asymptotically equal to $cn^{1/2}$ with $c = \pi\sqrt{2/3} = 2.5651\ldots$.

Now we study the EIPP. Let

$$\begin{aligned} I(x) &= \int_0^x \frac{t}{e^t - 1}\,dt, \\ \varphi(x) &= \frac{2I(x)^{2/3}}{x^{1/3}} - \frac{x^{2/3}}{I(x)^{1/3}} \ln(1 - e^{-x}), \\ C &= \max\{\varphi(x) : x \in \mathbb{R}_+\} = 2.2324\ldots, \\ D &= \frac{1}{3}\sqrt[3]{6} = 0.6057\ldots. \end{aligned}$$

Theorem 4. *We have for the EIPP*

$$Cn^{1/3} \lesssim \ln M(n) \lesssim Dn^{1/3} \ln n \text{ as } n \to \infty.$$

5.1. Proof of the lower bound in Theorem 4

Let $p(n,k,h)$ (resp. $P(n,k,h)$) be the number of partitions of the integer n into exactly (resp. at most) k parts, each less or equal to h, i.e. the number of representations of n as a sum of k integers

$$n = a_1 + \cdots + a_k,$$

where $1 \le a_1 \le \cdots \le a_k \le h$ (resp. $0 \le a_1 \le \cdots \le h$). If there is no restriction on the size of the parts we write briefly $p(n,k)$ (resp. $P(n,k)$).

Let

$$\begin{aligned} \nu &= \operatorname{argmax}\{\varphi(x) : x \in \mathbb{R}_+\}, \\ \mu &= \frac{\nu}{I(\nu)^{1/2}}, \\ \kappa &= \mu^{2/3}. \end{aligned}$$

Note that $C = \varphi(\nu)$. Let $k = \lfloor \kappa n^{1/3} \rfloor$ and $h = \lfloor \frac{n-1}{k} \rfloor$. Further let

$$g_i = \begin{cases} i & \text{if } i \leq h, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $M(n) \geq N(\boldsymbol{g})$. It is sufficient to show that $\ln N(\boldsymbol{g}) \sim Cn^{1/3}$. Here and in the following all asymptotic estimates are given for $n \to \infty$.

If $\boldsymbol{\lambda}$ is an admissible solution of the EIPP then

$$\sum_{i=1}^{n} g_i \lambda_i \begin{cases} = n & \text{if } \lambda_i = 0 \text{ for all } i > h, \\ < n & \text{otherwise.} \end{cases}$$

Hence $\boldsymbol{\lambda}$ has maximal gain iff $\lambda_i = 0$ for all $i > h$. For these vectors $\boldsymbol{\lambda}$ we have

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{h} \lambda_i \geq \frac{1}{h} \sum_{i=1}^{h} i\lambda_i = \frac{n}{h} \geq \frac{n}{\frac{n-1}{k}} > k.$$

Consequently, $\boldsymbol{\lambda} \in \mathbb{N}^n$ is an optimal solution of the EIPP iff

$$\sum_{i=1}^{n} i\lambda_i = n, \;\; \lambda_i = 0 \text{ for all } i > h \text{ and } \sum_{i=1}^{n} \lambda_i = k+1,$$

which implies that

$$N(\boldsymbol{g}) = p(n, k+1, h). \tag{10}$$

We have

$$p(n, k+1, h) = P((k+1)h - n, k+1, h-1) \tag{11}$$

since with each partition $n = a_1 + a_2 + \cdots + a_{k+1}$ counted by $p(n, k+1, h)$ we may associate bijectively the partition $(h - a_{k+1}) + \cdots + (h - a_2) + (h - a_1)$ counted by $P((k+1)h - n, k+1, h-1)$.

Let r be the remainder of the division $\frac{n-1}{k}$, $0 \leq r < k$. Then $(k+1)h - n = h - (r+1)$. Let $n' = h - (r+1)$ and $k' = k+1$. Since $n' \leq h - 1$ we have

$$P(n', k', h-1) = P(n', k') \tag{12}$$

and (10) - (12) imply

$$N(\boldsymbol{g}) = P(n', k'). \tag{13}$$

From a result of Szekeres [8, 9] which was reproved in a more accessible way by Canfield [10] it follows that

$$\ln P(n', k') \sim \sqrt{n'} g(u), \tag{14}$$

where $u = \frac{k'}{\sqrt{n'}}$, $g(u) = \frac{2v}{u} - u \ln(1 - e^{-v})$ and v is determined implicitly by $u = \frac{v}{I(v)^{1/2}}$.

We have $k' \sim \kappa n^{1/3}$, $n' \sim h \sim \frac{1}{\kappa} n^{2/3}$ and hence $u \sim \mu$, $v \sim \nu$ and

$$\sqrt{n'} g(u) \sim \frac{1}{\kappa^{1/2}} \left(\frac{2\nu}{\mu} - \mu \ln(1 - e^{-\nu}) \right) n^{1/3} = \varphi(\nu) n^{1/3}. \tag{15}$$

The relations (13) - (15) together with $C = \varphi(\nu)$ finally give

$$\ln N(\boldsymbol{g}) \sim C n^{1/3}.$$

□

5.2. Proof of the upper bound in Theorem 4

Lemma 4. *Let $\boldsymbol{\lambda}$ be an optimal solution of the EIPP. Then*

$$|\operatorname{supp}(\boldsymbol{\lambda})| \leq \sqrt[3]{6n}.$$

Proof. Assume that there is an optimal solution $\boldsymbol{\lambda}$ of the EIPP such that $|\operatorname{supp}(\boldsymbol{\lambda})| > \sqrt[3]{6n}$. Let $s = |\operatorname{supp}(\boldsymbol{\lambda})|$ and $\operatorname{supp}(\boldsymbol{\lambda}) = \{a_1, \dots, a_s\}$ with $0 < a_1 < \dots < a_s < n$. Let $0 < k \leq s$. We consider the subset

$$S^k_{0,0} := \{a_1, \dots, a_k\}$$

and the following $k(s-k)$ subsets $S^k_{i,j}$ of $\operatorname{supp}(\boldsymbol{\lambda})$:

$$S^k_{i,j} := \{a_1, \dots, a_{k-i}, a_{k-i+j+1}, a_{s-i+2}, \dots, a_s\}, \quad i = 1, \dots, k,\ j = 1, \dots, s-k.$$

Let $\boldsymbol{x}^k_{i,j}$ be the characteristic vector of $S^k_{i,j}$, $(i,j) \in \{(0,0)\} \cup ([k] \times [s-k])$. Clearly, for all $(i,j) \in \{(0,0)\} \cup ([k] \times [s-k])$

$$\mathbf{0} \leq \boldsymbol{x}^k_{i,j} \leq \boldsymbol{\lambda} \text{ and } \mathbf{1} \cdot \boldsymbol{x}^k_{i,j} = k.$$

Moreover,

$$1 \leq \overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{x}^k_{0,0} < \overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{x}^k_{1,1} < \dots < \overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{x}^k_{1,s-k} < \overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{x}^k_{2,1} < \dots < \overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{x}^k_{k,s-k} \leq n.$$

Hence we have for each k exactly $k(s-k)+1$ different integers of the form $\overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{x}$ between 1 and n. With $k = 1, \ldots, s$ this gives in total

$$\sum_{k=1}^{s} (k(s-k)+1) = \frac{s}{6}(s^2+5) > \frac{s^3}{6} > n$$

integers between 1 and n. Hence two of them are equal, say $\overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{x}$ with $\boldsymbol{x} = \boldsymbol{x}_{i,j}^{k}$ and $\overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{x}'$ with $\boldsymbol{x}' = \boldsymbol{x}_{i',j'}^{k'}$, where $0 < k < k' \leq s$. Then

$$\overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{x} = \overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{x}' \text{ and } k = \mathbf{1} \cdot \boldsymbol{x} < \mathbf{1} \cdot \boldsymbol{x}' = k'. \tag{16}$$

Let

$$\boldsymbol{\lambda}^{+} = \boldsymbol{\lambda} - \boldsymbol{x} + \boldsymbol{x}' \text{ and } \boldsymbol{\lambda}^{-} = \boldsymbol{\lambda} - \boldsymbol{x}' + \boldsymbol{x}. \tag{17}$$

Then $\boldsymbol{\lambda}^{+}$ and $\boldsymbol{\lambda}^{-}$ are different admissible solutions of the EIPP. According to $\boldsymbol{\lambda} = \frac{1}{2}(\boldsymbol{\lambda}^{+} + \boldsymbol{\lambda}^{-})$, both admissible solutions are in fact optimal and hence $\mathbf{1} \cdot \boldsymbol{\lambda}^{+} = \mathbf{1} \cdot \boldsymbol{\lambda}^{-}$, which leads with (17) to $\mathbf{1} \cdot \boldsymbol{x} = \mathbf{1} \cdot \boldsymbol{x}'$, a contradiction to (16). □

For a set $S = \{a_1, \ldots, a_s\}$ of s positive integers, let $p_S(n)$ be the number of partitions of the integer n with support S, i.e.

$$p_S(n) = |\{\boldsymbol{\lambda} \in \mathbb{N}^n : \overrightarrow{\boldsymbol{n}} \cdot \boldsymbol{\lambda} = n \text{ and } \operatorname{supp}(\boldsymbol{\lambda}) = S\}|.$$

We use the following rough upper bound for $p_S(n)$:

Lemma 5. *Let $S = \{a_1, \ldots, a_s\}$ be a set of s positive integers. Then*

$$p_S(n) \leq \frac{n^s}{s!} \prod_{i=1}^{s} \frac{1}{a_i}.$$

Proof. Let $\boldsymbol{a} = (a_1, \ldots, a_s)$ and $\Lambda_{\boldsymbol{a}} = \{\boldsymbol{\lambda} \in [n]^s : \boldsymbol{a} \cdot \boldsymbol{\lambda} = n\}$. Then

$$p_S(n) = |\Lambda_{\boldsymbol{a}}|.$$

With each $\boldsymbol{\lambda} \in \Lambda_{\boldsymbol{a}}$ we associate the half-open cube $[\lambda_1 - 1, \lambda_1) \times \cdots \times [\lambda_s - 1, \lambda_s)$. Note that the cubes of different elements of $\Lambda_{\boldsymbol{a}}$ are disjoint. Moreover, all such cubes are contained in the simplex $\{\boldsymbol{x} \in \mathbb{R}_{+}^{s} : \boldsymbol{a} \cdot \boldsymbol{x} \leq n\}$. Since each cube has volume 1 and the simplex has volume $\frac{1}{s!} \prod_{i=1}^{s} \frac{n}{a_i}$ the bound follows. □

Lemma 6. *The number of partitions of the integer n with support size s can be bounded as follows:*

$$\sum_{1\le a_1<\cdots<a_s\le n} p_{\{a_1,\ldots,a_s\}}(n) \le \left(\frac{e^2 n(1+\ln n)}{s^2}\right)^s.$$

Proof. Let $A(n)$ be the set of all ordered s-tuples $(a_1,\ldots,a_s)$ of distinct elements of $[n]$. We have by Lemma 5 and in view of $s! \ge (s/e)^s$

$$\sum_{1\le a_1<\cdots<a_s\le n} p_{\{a_1,\ldots,a_s\}}(n) \le \frac{1}{s!}\sum_{(a_1,\ldots,a_s)\in A(n)} p_{\{a_1,\ldots,a_s\}}(n) \le$$

$$\frac{1}{s!}\sum_{(a_1,\ldots,a_s)\in A(n)} \frac{n^s}{s!}\prod_{i=1}^{s}\frac{1}{a_i} \le \frac{n^s}{(s!)^2}\sum_{1\le a_1,\ldots,a_s\le n}\prod_{i=1}^{s}\frac{1}{a_i}$$

$$= \frac{n^s}{(s!)^2}\left(\sum_{i=1}^{s}\frac{1}{i}\right)^s \le \frac{n^s}{(s!)^2}(1+\ln n)^s \le \left(\frac{e^2 n(1+\ln n)}{s^2}\right)^s.$$

□

Now the proof of the upper bound in Theorem 4 can be completed as follows: According to Lemma 4, the number $M(n)$ can be bounded from above by the number of partitions of the integer n with support size at most $(6n)^{(1/3)}$ and using Lemma 6 one obtains for sufficiently large n

$$M(n) \le \sum_{s\le(6n)^{(1/3)}}\left(\frac{e^2 n(1+\ln n)}{s^2}\right)^s \le 2\left(\frac{e^2 n(1+\ln n)}{(6n)^{(2/3)}}\right)^{(6n)^{(1/3)}}$$

and hence

$$\ln M(n) \lesssim (6n)^{(1/3)}\frac{1}{3}\ln n.$$

□

6. Open problems

Problem 1. *Does there exist an algorithm of time complexity $O(n\ln(n))$ for the solution of the IPP/EIPP?*

Problem 2. *Does there exist a polynomial-time algorithm for the determination of an admissible/optimal solution of the IPP/EIPP with minimum support size?*

Problem 3. *Improve the bounds in Theorem 4.*

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