COMPLETENESS AND COMPACTNESS CRITERIA FOR $\mathbb{R}$-TREES

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Abstract. We establish (geometric) criteria for an $\mathbb{R}$-tree to be complete, compact, or locally compact. As a byproduct, we obtain a generalization of a result of Wilkens concerning bounded group actions on $\mathbb{R}$-trees.

1. Introduction

Given a simplicial tree there is an integer-valued metric, the so-called path metric, on the set of vertices (the distance between two vertices being the number of edges in a shortest path joining them). An $\mathbb{R}$-tree is a metric space whose metric has certain properties in common with the path metric. More generally, one can define Λ-trees, where Λ is an arbitrary ordered abelian group (in this case the metric takes values in Λ); the standard references for this theory are [1] and [2]. See [6] and [7] for a very readable overview.

Part of the original definition by Tits [8] of an $\mathbb{R}$-tree was that it was a complete metric space. However, for many purposes, this assumption is unnecessary, and today completeness is no longer assumed as part of the definition. In fact, the metric completion of an $\mathbb{R}$-tree is also an $\mathbb{R}$-tree, an observation first made by Imrich [5]. There are several ways of defining an $\mathbb{R}$-tree, discussed in Chapter 2, §2 of [2]. Without the assumption of completeness, Tits' definition is that it is a geodesic metric space having no subspace homeomorphic to a circle (see Proposition 2.3(3), Chapter 2 in [2]). Imrich in effect used the criterion that an $\mathbb{R}$-tree is a 0-hyperbolic geodesic metric space (see Lemmas 1.6 and 4.3, with $\Lambda = \mathbb{R}$, in Chapter 2 of [2]). However, he used an equivalent version of 0-hyperbolic (the 4-point condition); see the discussion after Lemma 2.6, Chapter 1 in [2].

The first purpose of the present note is to give a more explicit description of the completion of an $\mathbb{R}$-tree, resulting in a criterion for an $\mathbb{R}$-tree to be complete (Theorem 1 below). This result is perhaps well-known to those working on $\mathbb{R}$-trees, but we know of no proof in the literature, and it is not a triviality. An action by isometries of a group G on a metric space (X, d) is said to be *bounded* if for some point $x_0 \in X$, there is a positive constant K such that $d(x_0, gx_0) \leq K$ for all $g \in G$. (If this is true for some point x_0, it is true for all points in X by a simple application of the triangle inequality.) Wilkens [9] showed that if a group has a bounded action on a complete $\mathbb{R}$-tree, then it has a fixed point; cf. also Lemma 2.5 in [2, Chapter 4]. It follows from our results that, if a group has a bounded action on an $\mathbb{R}$-tree, then either there is a fixed point or there is a fixed open end whose rays are bounded. (This last idea will be explained later.)

In Section 3, building on Theorem 1, we provide a criterion for an $\mathbb{R}$-tree to be compact (Theorem 2), which, when coupled with Theorem 1, is expressed entirely in terms of the geometry of the tree. We finish with a geometric criterion for an $\mathbb{R}$-tree to be locally compact.

2. Complete $\mathbb{R}$-trees

Let (X, d) be an $\mathbb{R}$-metric space, and let $\mathcal{C} = \mathcal{C}(X)$ denote the set of Cauchy sequences in X. Define a binary relation on $\mathcal{C}$ via

$$(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}} \iff \lim_{i\to\infty} d(x_i, y_i) = 0.$$

This is an equivalence relation on $\mathcal{C}$, and we recall that, as a set, the completion $\hat{X}$ of X is just $\mathcal{C}/\sim$, the set of equivalence classes. When there is no ambiguity, we shall write simply (x_i) instead of $(x_i)_{i\in\mathbb{N}}$, and we shall denote the equivalence class of (x_i) by $\langle x_i\rangle$. We also recall that the metric $\hat{d}$ on $\hat{X}$ is given by

$$\hat{d}(\langle x_i\rangle, \langle y_i\rangle) = \lim_{i\to\infty} d(x_i, y_i), \quad (x_i), (y_i) \in \mathcal{C}.$$

As in [2], we use $[x, y]$ to denote the segment joining points x, y in a Λ-tree.

Definition 1. Let Λ be a totally ordered abelian group, and let (X, d) be a Λ-tree with base point x_0. A sequence (y_i) in X is called *monotone increasing* (with respect to x_0), if $i < j$ implies $y_i \in [x_0, y_j]$.

The crucial fact needed to describe the completion of an $\mathbb{R}$-tree and to characterise complete $\mathbb{R}$-trees is the following.

Proposition 1. *Let (X, d) be an $\mathbb{R}$-tree, let x_0 be a base point in X, and suppose that (y_i) is a Cauchy sequence in X. Then there is a monotone increasing Cauchy sequence (z_i) in X with respect to x_0, such that $(z_i) \sim (y_i)$.*

Proof. Given points $x,\ y \in X$, denote by $x \wedge y$ the unique point z such that

$$[x_0, x] \cap [x_0, y] = [x_0, z],$$

so that $d(x_0, z) = (x \cdot y)_{x_0}$, where

$$(x \cdot y)_{x_0} = \frac{1}{2}\big\{d(x, x_0) + d(y, x_0) - d(x, y)\big\};$$

see Lemma 1.2 in [2, Chapter 2] and the discussion following it. Set $\|x\| := d(x, x_0)$. Then

$$\|x \wedge y\| = (x \cdot y)_{x_0} \leq \min\big\{\|x\|, \|y\|\big\}.$$

We have

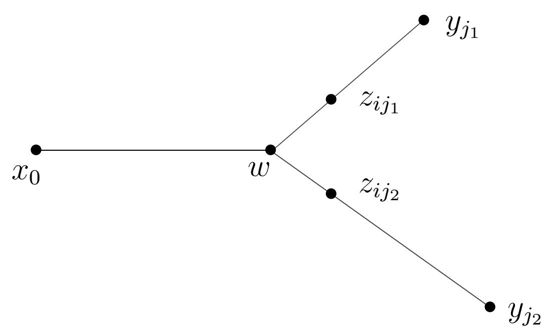
$$\begin{aligned} d(y_i, y_j) &= \|y_i\| + \|y_j\| - 2(y_i, y_j)_{x_0} \\ &= \|y_i\| + \|y_j\| - 2\|y_i \wedge y_j\| \\ &\geq \|y_i\| + \|y_j\| - 2\min(\|y_i\|, \|y_j\|) \\ &= \big|\|y_i\| - \|y_j\|\big|, \end{aligned}$$

hence $(\|y_i\|)$ is a Cauchy sequence in $\mathbb{R}$, so $\|y_i\| \to \alpha$ with $\alpha \in \mathbb{R}$, since $\mathbb{R}$ with the usual metric is complete.

If $\alpha = 0$, we take $z_i = x_0$ for all i. Then (z_i) is clearly a a monotone increasing Cauchy sequence with $(z_i) \sim (y_i)$. Otherwise, $\alpha > 0$, and we can choose a sequence (α_i) in $\mathbb{R}$, which is monotone increasing, converges to α, and satisfies $0 \leq \alpha_i < \alpha$ for all i. Fix an index i, and define z_{ij} to be the unique point in X satisfying $z_{ij} \in [x_0, y_j]$ and $\|z_{ij}\| = \min\{\alpha_i, \|y_j\|\}$. By choice of the sequence (α_i) and the fact that $\|y_j\| \to \alpha$, we have $\|y_j\| \geq \alpha_i$ for j sufficiently large, so there exists j_0 depending only on i, such that $\|z_{ij}\| = \alpha_i$ for $j \geq j_0$. Let $j_1, j_2 \geq j_0$ be indices such that $z_{ij_1} \neq z_{ij_2}$, and set $w = z_{ij_1} \wedge z_{ij_2}$. We have $z_{ij_1} \neq w \neq z_{ij_2}$, because $\|z_{ij_1}\| = \|z_{ij_2}\|$. Also, $w \in [x_0, z_{ij_1}] \subseteq [x_0, y_{j_1}]$, so $z_{ij_1} \in [w, y_{j_1}]$, hence $[y_{j_1}, z_{ij_1}] \cap [z_{ij_1}, w] = \{z_{ij_1}\}$. Similarly, $[y_{j_2}, z_{ij_2}] \cap [z_{ij_2}, w] = \{z_{ij_2}\}$. Also, $[z_{ij_1}, w] \cap [w, z_{ij_2}] = \{w\}$ (see Lemma 1.2(1) in [2, Chapter 2]). It follows from Lemma 1.5 in [2, Chapter 2] that

$$[y_{j_1}, y_{j_2}] = [y_{j_1}, z_{ij_1}, w, z_{ij_2}, y_{j_2}]$$

in the notation established after Corollary 1.3 in [2, Chapter 2]. The situation is illustrated in the following diagram.



By Lemma 1.4 in [2, Chapter 2],

$$\begin{aligned} d(y_{j_1}, y_{j_2}) &= d(y_{j_1}, z_{ij_1}) + d(z_{ij_1}, w) + d(z_{ij_2}, w) + d(y_{j_2}, z_{ij_2}) \\ &\geq d(y_{j_1}, z_{ij_1}) + d(y_{j_2}, z_{ij_2}) \\ &= (\|y_{j_1}\| - \alpha_i) + (\|y_{j_2}\| - \alpha_i). \end{aligned}$$

Since $\alpha_i < \alpha$, we can choose an element $\varepsilon_i \in \mathbb{R}$, such that $0 < 2\varepsilon_i \leq \alpha - \alpha_i$. Since $\|y_j\| \to \alpha$, there exists $j_0' = j_0'(i)$ such that $|\alpha - \|y_j\|| < \varepsilon_i$ for $j \geq j_0'$. Then, for $j \geq j_0'$,

$$\|y_j\| - \alpha_i = (\|y_j\| - \alpha) + (\alpha - \alpha_i) > -\varepsilon_i + 2\varepsilon_i = \varepsilon_i.$$

Hence,

$$d(y_{j_1}, y_{j_2}) > 2\varepsilon_i \quad \text{for } j_1, j_2 \geq \max\{j_0, j_0'\}.$$

Since the sequence $(y_j)_{j\in\mathbb{N}}$ has the Cauchy property, the last estimate implies that there exists some j_0'' such that at least one of j_1, j_2 is less than j_0''. Hence, the sequence $(z_{ij})_{j\in\mathbb{N}}$ is ultimately constant, and we define z_i to be equal to this constant, so that $\|z_i\| = \alpha_i$.

Given natural numbers i_1, i_2 with $i_1 < i_2$, we have $\alpha_{i_1} \leq \alpha_{i_2}$ (since the sequence $(\alpha_i)_{i\in\mathbb{N}}$ is monotone increasing), and, by our construction, there exists $j \in \mathbb{N}$ such that $z_{i_1} = z_{i_1 j}$ and $z_{i_2} = z_{i_2 j}$, and we have $\|z_{i_1}\| = \alpha_{i_1}$, $\|z_{i_2}\| = \alpha_{i_2}$. Consequently, z_{i_1} and z_{i_2} both lie on the segment $[x_0, y_j]$, and we have $\|z_{i_1}\| \leq \|z_{i_2}\|$, so $z_{i_1} \in [x_0, z_{i_2}]$; that is, the sequence $(z_i)_{i\in\mathbb{N}}$ is monotone increasing with respect to x_0. Moreover, we have

$$d(z_i, z_j) = |\alpha_i - \alpha_j| \to 0 \qquad \text{as } i, j \to \infty,$$

hence (z_i) is a Cauchy sequence in X.

Let $\varepsilon \in \mathbb{R}$ be a given positive element, and choose $k = k(\varepsilon)$ such that $i \geq k$ implies $\alpha - \alpha_i < \varepsilon/3$ and $i,\ j \geq k$ implies $d(y_i, y_j) < \varepsilon/3$. Moreover, given i, there exists $\ell = \ell(\varepsilon, i)$ such that $j \geq \ell$ implies $\|y_j\| - \alpha < \varepsilon/3$ as well as $z_{ij} = z_i$, so that $z_i \in [x_0, y_j]$. Now, for $i \geq k(\varepsilon)$ and $j \geq \max\{k(\varepsilon), \ell(\varepsilon, i)\}$,

$$\begin{aligned} d(z_i, y_i) &\leq d(z_i, y_j) + d(y_i, y_j) \\ &= (\|y_j\| - \|z_i\|) + d(y_i, y_j) \\ &= (\|y_j\| - \alpha) + (\alpha - \alpha_i) + d(y_i, y_j) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon. \end{aligned}$$

Thus $i \geq k$ implies $d(z_i, y_i) < \varepsilon$, so $\lim_{i\to\infty} d(z_i, y_i) = 0$, that is, $(y_i) \sim (z_i)$. □

If (X, d) is a metric space, recall that X embeds isometrically in $\hat{X}$ by mapping $x \in X$ to the equivalence class $\hat{x} = \langle x \rangle$ of the constant sequence $x_i = x$. This equivalence class is the set of all Cauchy sequences in X which converge to x. We shall identify x with $\hat{x}$.

We shall later use the ideas of *degree* of a point, branch point and endpoint; see the definition after Lemma 1.7, Ch.2 in [2]. The definition of *end*, *ray* and the notation $[x, \varepsilon\rangle$, where x is a point and ε is an end of a Λ-tree are explained in §3 of [2, Chapter 2]. It is shown in [2, Ch.2, Lemma 3.4] that rays are closed subtrees, meaning they are convex closed. In an $\mathbb{R}$-tree, closed subtrees are in fact closed in the metric topology. For suppose A is a closed subtree, and $b \notin A$. Let a be the projection of b onto A (see the discussion after Lemma 1.9 in [2, Chapter 2]). Then $d(x, b) \geq d(a, b) > 0$ for all $x \in A$, so no sequence in A can converge to b.

If ε is an end of an $\mathbb{R}$-tree (X, d) and $x,\ y \in X$, then the rays $[x, \varepsilon\rangle$ and $[y, \varepsilon\rangle$ intersect in a ray $[z, \varepsilon\rangle$, and $[x, \varepsilon\rangle = [x, z] \cup [z, \varepsilon\rangle$, $[y, \varepsilon\rangle = [y, z] \cup [z, \varepsilon\rangle$ (see the remarks after Lemma 3.5 in [2, Chapter 2]). Hence $[x, \varepsilon\rangle$ is bounded if, and only if, $[y, \varepsilon\rangle$ is (i.e. if, and only if, $[z, \varepsilon\rangle$ is bounded). In this situation, we say that ε *has bounded rays.*

Definition 2. Let (X, d) be an $\mathbb{R}$-tree. We denote by $\mathcal{BO}(X)$ the set of all open ends of X with bounded rays. (This set may be empty.)

Let (X, d) be an $\mathbb{R}$-tree. We extend the identity map on X to a map $\varphi : \hat{X} \to X \cup \mathcal{BO}(X)$ as follows. An element y of $\hat{X} \setminus X$ is an equivalence class of Cauchy sequences which do not converge in X. Choose a base point $x_0 \in X$. Then by Proposition 1, there is a monotone increasing sequence (z_i) in the equivalence class y, and it does not converge. Let $\alpha_i = d(x_0, z_i)$. Then (α_i) is a bounded increasing sequence in $\mathbb{R}$, and so converges, to $\sup\{\alpha_i\} = \alpha$, say. Let

$$L = \bigcup_{i \in \mathbb{N}} [x_0, z_i].$$

This set is the union of an increasing sequence of segments with x_0 as an endpoint; it follows easily that it is a linear subtree of X with x_0 as an endpoint. Also, L is bounded as $d(x_0, y) \leq \alpha$ for all $y \in L$.

If L is not a ray, or if it is a ray corresponding to a closed end of X, then $L \subseteq [x_0, y]$ for some $y \in X$, and $d(x_0, z_i) \leq d(x_0, y)$ for all i. Hence $d(x_0, y) \geq \alpha_i$ for all i, so $d(x_0, y) \geq \alpha$. Therefore, there is a point $z \in [x_0, y]$ such that $d(x_0, z) = \alpha$. But then $d(z_i, z) = \alpha - \alpha_i \to 0$ as $i \to \infty$, so (z_i) converges to z, a contradiction. Thus, L defines an open end ε of X with bounded rays, and we set $\varphi(y) = \varepsilon$.

Lemma 1. *The mapping $\varphi : \hat{X} \to X \cup \mathcal{BO}(X)$ is a well-defined bijection, and is independent of the choice of the base point x_0.*

Proof. Suppose that (z_i), (z'_i) are monotone increasing sequences, with respect to base points x_0, x'_0, respectively, which do not converge in X. These sequences define open ends ε, ε' as explained above, with $[x_0, \varepsilon\rangle = \bigcup_{i \in \mathbb{N}} [x_0, z_i]$ and $[x'_0, \varepsilon'\rangle = \bigcup_{i \in \mathbb{N}} [x'_0, z'_i]$.

Assuming $(z_i) \sim (z'_i)$, we show that $\varepsilon = \varepsilon'$. This will demonstrate that the map φ is well-defined and independent of the choice of a base point.

First, we have $[x_0, \varepsilon\rangle \cap [x'_0, \varepsilon'\rangle \neq \emptyset$, for otherwise $d(z_i, z'_i) \geq d([x_0, \varepsilon\rangle, [x'_0, \varepsilon'\rangle) > 0$ for all i, contradicting the fact that $(z_i) \sim (z'_i)$. (The definition of $d(A, B)$, where A, B are closed subtrees, is given after Lemma 1.9 in Chapter 2 of [2].) Choose a point y in this intersection, and suppose for a contradiction that $\varepsilon \neq \varepsilon'$. By Lemma 3.7 in [2, Chapter 2], the intersection $[y, \varepsilon\rangle \cap [y, \varepsilon'\rangle$ contains a point x such that $[x, \varepsilon\rangle \cap [x, \varepsilon'\rangle = \{x\}$. Now $x \in [x_0, \varepsilon\rangle$, so $x \in [x_0, z_k]$ for some k, and since (z_i) is monotone increasing with respect to x_0, $z_i \in [x, \varepsilon\rangle$ for $i \geq k$. Further, since (z_i) does not converge, we can choose k such that $z_k \neq x$. Similarly we can find k' such that, for $i \geq k'$, $z'_i \in [x, \varepsilon'\rangle$ and $z'_{k'} \neq x$. Then, for $i \geq \max\{k, k'\}$,

$$d(z_i, z'_i) = d(z_i, x) + d(x, z'_i) \geq d(z_k, x) + d(x, z'_{k'}) > 0$$

because (z_i) and (z'_i) are monotone increasing and $[x, \varepsilon\rangle \cap [x, \varepsilon'\rangle = \{x\}$. This contradicts our assumption that $(z_i) \sim (z'_i)$.

For the rest of the proof we can fix a base point x_0. We show that φ is injective. Suppose that (z_i), (z'_i) are monotone increasing sequences with respect to x_0, which do not converge in X and define the same open end ε in the way explained above, so that

$$\bigcup_{i \in \mathbb{N}} [x_0, z_i] = [x_0, \varepsilon\rangle = \bigcup_{i \in \mathbb{N}} [x_0, z'_i].$$

As in the proof of Proposition 1, we have $\|z_i\| \to \alpha$ and $\|z'_i\| \to \beta$ as $i \to \infty$, for some real numbers α, β. We claim that $\alpha = \beta$. Suppose not, so, without loss of generality, we can assume that $\alpha < \beta$. Since (z_i) is monotone increasing, $\|z_i\| \leq \alpha$ for all $i \in \mathbb{N}$, and since $[x_0, \varepsilon\rangle = \bigcup_{i \in \mathbb{N}} [x_0, z_i]$, we have $\|x\| \leq \alpha$ for all $x \in [x_0, \varepsilon\rangle$. Furthermore, since (z'_i) is monotone increasing, we can find z'_i such that $\alpha < \|z'_i\| \leq \beta$. But $z'_i \in [x_0, \varepsilon\rangle$, so this is a contradiction, implying that $\alpha = \beta$. Therefore,

$$d(z_i, z'_i) = \big| \|z_i\| - \|z'_i\| \big| \to |\alpha - \alpha| = 0$$

as $i \to \infty$, hence $(z_i) \sim (z'_i)$. Thus, φ is injective.

To show that φ is surjective, suppose ε is an open end of X with bounded rays. Then there is an isomorphism of metric spaces $\mu : [0, \alpha)_{\mathbb{R}} \to [x_0, \varepsilon\rangle$, for some positive real number α, with $\mu(0) = x_0$. Let (α_i) be an increasing sequence in $[0, \alpha)_{\mathbb{R}}$ converging to α and let $z_i = \mu(\alpha_i)$. Then $\|z_i\| = \alpha_i$ and $d(z_i, z_j) = |\alpha_i - \alpha_j|$, so (z_i) is a monotone increasing Cauchy sequence in X; suppose it converges in X, to z, say. Now $[x_0, \varepsilon\rangle$ is a closed subtree, so $z \in [x_0, \varepsilon\rangle$. Also,

$$d(x_0, z) = \lim_{i\to\infty} d(x_0, z_i) = \lim_{i\to\infty} \alpha_i = \alpha.$$

But all points of $[x_0, \varepsilon\rangle$ are at distance less than α from x_0, so this is a contradiction. Hence (z_i) does not converge. Clearly $[x_0, \varepsilon\rangle = \bigcup_{i\in\mathbb{N}}[x_0, z_i]$, so $\varepsilon = \varphi(\langle z_i\rangle)$ and φ is surjective. □

We can now state our criterion for completeness of $\mathbb{R}$-trees.

Theorem 1. *Let (X, d) be an $\mathbb{R}$-tree. Then the following assertions are equivalent.*

(i) *(X, d) is complete.*
(ii) *Every monotone increasing Cauchy sequence with respect to some base point converges.*
(iii) *(X, d) has no open ends with bounded rays.*

Proof. Obviously (i) ⇒ (ii). If $(z_i) \sim (y_i)$ and (z_i) converges, then (y_i) converges to the same point; thus, the implication (ii) ⇒ (i) follows from Proposition 1. Since (X, d) is complete if, and only if, $X = \hat{X}$, equivalence of (i) and (iii) follows by Lemma 1. □

Suppose a group G acts by isometries on an $\mathbb{R}$-tree (X, d). There is an induced action by isometries on the completion $\hat{X}$, given by $g\langle x_i\rangle = \langle gx_i\rangle$ for $g \in G$. There is also an induced action on the set of ends of X; if L is a ray defining an end ε, then $g\varepsilon$ is the end defined by the ray gL, for $g \in G$. This clearly restricts to an action on $\mathcal{BO}(X)$.

Lemma 2. *If a group G acts by isometries on an $\mathbb{R}$-tree (X, d), then, with the induced actions, the mapping $\varphi : \hat{X} \to X \cup \mathcal{BO}(X)$ is G-equivariant.*

Proof. Let (z_i) be a monotone increasing Cauchy sequence with respect to a base point x_0 which does not converge, and let $\varphi(\langle z_i\rangle) = \varepsilon$, so that $\bigcup_{i\in\mathbb{N}}[x_0, z_i] = [x_0, \varepsilon\rangle$. Let $g \in G$. Then (gz_i) is a monotone increasing Cauchy sequence with respect to gx_0 which does not converge. The ray $g[x_0, \varepsilon\rangle$ defines $g\varepsilon$ and equals $\bigcup_{i\in\mathbb{N}}[gx_0, gz_i]$. Hence $g\varepsilon = \varphi(\langle gz_i\rangle)$ (using the fact that the definition of φ does not depend on the choice of base point). That is, $g\varphi(\langle z_i\rangle) = \varphi(g\langle z_i\rangle)$, as required. □

Wilkens [9] has shown that if a group G has a bounded action by isometries on a complete $\mathbb{R}$-tree, then there is a fixed point. Now, if G has a bounded action on an $\mathbb{R}$-tree, then the induced action on the completion is easily seen to be bounded as well. Lemmas 1 and 2 therefore give the following.

Proposition 2. *If a group has a bounded action by isometries on an $\mathbb{R}$-tree, then either there is a fixed point, or there is a fixed open end with bounded rays.* □

Let (X, d) be an $\mathbb{R}$-tree with base point x_0. The completion of X is obtained by adding a point for every open end ε with bounded rays. The new point is of the form $z = \langle z_i\rangle$ where z is a monotone increasing Cauchy sequence with respect to x_0 which does not converge in X, $\varphi(z) = \varepsilon$, and $[x_0, \varepsilon\rangle = \bigcup_{i\geq 1}[x_0, z_i]$. If $m \geq n$, then $d(x_0, z_m) = d(x_0, z_n) + d(z_n, z_m)$, and letting $m \to \infty$, $d(x_0, z) = d(x_0, z_n) + d(z_n, z)$, so

$z_n \in [x_0, z]$ in $\hat{X}$. Hence $[x_0, \varepsilon\rangle \subseteq [x_0, z]$. In fact, since $d(z_n, z) \to 0$ as $n \to \infty$, we have that $[x_0, z] = [x_0, \varepsilon\rangle \cup \{z\}$. It follows from the next lemma that z is an endpoint of $\hat{X}$. Thus $\hat{X}$ is obtained by adding an endpoint to the corresponding ray, for each open end of X with bounded rays. This next lemma will also be useful later.

Lemma 3. *Let (X, d) be an $\mathbb{R}$-tree and let Y be a dense subset of X. Choose a base point $x_0 \in X$. Then*

(i) *If $x \in X$ and x is not an endpoint of X, then $x \in [x_0, y]$ for some $y \in Y$.*
(ii) *If Y is a subtree of X, then every point of $X \setminus Y$ is an endpoint of X.*

Proof. (i) We may assume $x_0 \neq x$, otherwise any point y in Y will do. Then $[x, x_0]$ defines a direction at x, and there is at least one more, defined by $[x, z]$ say, where $z \in X$. Then $x \in [x_0, z]$. The open ball $B(z, d(z, x))$ contains a point of Y, say y. The projection p of y onto $[x_0, z]$ lies in $[x, z]$ and is not equal to x, otherwise

$$d(y, z) = d(y, p) + d(p, x) + d(x, z) \geq d(x, z)$$

contradicting the fact that $y \in B(z, d(z, x))$. By [2, Ch.2, Lemma 1.5], $[x_0, y] = [x_0, x, p, y]$, so $x \in [x_0, y]$.

(ii) We can choose $x_0 \in Y$. Then any point $x \in X$ which is not an endpoint is in $[x_0, y]$ for some $y \in Y$, and $[x_0, y] \subseteq Y$ whence (ii). □

3. Compact ℝ-trees

For the following two definitions, let (X, d) be an $\mathbb{R}$-tree, and let $r > 0$ be a real number.

Definition 1. Let $x \in X$ be a point.

(a) The direction at x *has length* $\geq r$, if it is represented by a segment of length $d(x, y) \geq r$.
(b) The $\mathbb{R}$-tree X is *r-locally finite at x,* if the set of all directions at x of length $\geq r$ is finite. It is *r-locally finite* if it is r-locally finite at x for all $x \in X$.

Definition 2. (a) A segment $[x, y]$ has *r-finite branching* if the set of points $p \in [x, y]$ such that there exists $z \in X$ with p being the projection of z onto $[x, y]$ and $d(z, p) = d(z, [x, y]) \geq r$, is finite.
(b) The $\mathbb{R}$-tree X itself has r-finite branching, if every segment in X has r-finite branching.

Theorem 2. *An $\mathbb{R}$-tree X is compact if, and only if,*

(i) *it is bounded,*
(ii) *it is complete,*
(iii) *it is r-locally finite for every $r > 0$,*
(iv) *it has r-finite branching for every $r > 0$.*

Proof. Since X is a metric space, it is enough to show that conditions (i)–(iv) are equivalent to sequential compactness. Further, it is well known that a compact metric space is complete and bounded. Suppose there exists $x_0 \in X$ and $r > 0$ such that the set of directions at x_0 of length $\geq r$ is infinite. Then there exists a sequence of points $(x_i)_{i\geq 1}$ in X such that $d(x_0, x_i) = r$ and $[x_0, x_i] \cap [x_0, x_j] = \{x_0\}$ for all $i, j \geq 1$ and $i \neq j$. Then $d(x_i, x_j) = 2r$ whenever $i \neq j$, thus no subsequence of $(x_i)_{i\geq 1}$ can converge. Hence, condition (iii) is also necessary for compactness. Suppose condition (iv) fails, so there is a segment $[x, y]$, a real number $r > 0$ and infinitely many distinct points p_n, $n \geq 1$ in $[x, y]$ which are the projections onto $[x, y]$ of points y_n with $d(y_n, p_n) \geq r$. Then $d(y_i, y_j) = d(y_i, p_i) + d(p_i, p_j) + d(y_j, p_j) \geq 2r$ whenever $i \neq j$, so no subsequence of (y_i) can converge. Hence, condition (iv) is also necessary for compactness.

Conversely, assume conditions (i)–(iv), and let (y_i) be a sequence in X. We have to show that (y_i) has a convergent subsequence. We can assume (y_i) has no constant subsequence, so replacing (y_i) by a subsequence, we can assume that

$$y_i \neq y_j \text{ for } i \neq j. \tag{1}$$

Suppose there is a point $x \in X$ such that the set of directions defined by the segments $[x, y_n]$, $n \geq 1$ is infinite. Then by condition (iii) we can find a subsequence of (y_i) converging to x. Thus we can assume that

$$\text{for any } x \in X, \text{ the set of directions defined by } \big\{[x, y_n] \mid n \geq 1,\ y_n \neq x\big\} \text{ is finite.} \tag{2}$$

Suppose x, $y \in X$ and let q_n be the projection of y_n onto $[x, y]$. If the set $\{q_n \mid n \geq 1\}$ is infinite, there is a subsequence (q_{i_j}) of distinct points converging to a point $q \in [x, y]$, since $[x, y]$ is compact. Using condition (iv), the subsequence (y_{i_j}) of (y_i) also converges to q. We can therefore assume that

$$\text{for any } x,\ y \in X, \text{ the set of projections of the points } y_n, \text{ for } n \geq 1, \text{ onto } [x, y] \text{ is finite.} \tag{3}$$

Now put $p_1 = z_1 = y_1$. By (1) and (2), there is a direction at z_1 defined by $[z_1, y_n]$ for infinitely many y_n. Choose n smallest such that $[z_1, y_n]$ defines this direction and put $z_2 = y_n$.

By (3), there are finitely many points in $[z_1, z_2]$ which are projections of at least one of the y_n onto it. Therefore, by (1) there exists $p_2 \in [z_1, z_2]$ such that $p_2 \neq z_1$ and p_2 is the projection onto $[z_1, z_2]$ of infinitely many of the points y_n, $n \geq 1$. (Note that $p_2 = z_2$ is possible.) By (1) and (3), infinitely many of the y_n have p_2 as their projection onto $[z_1, z_2] = [p_1, z_2]$ and define the same direction at p_2. Choose such a y_n with n as small as possible and put $z_3 = y_n$. Now as before we can find $p_3 \in [p_2, z_3]$ such that $p_3 \neq p_2$ and infinitely many of the y_n have p_3 as their projection onto $[p_2, z_3]$. Note that p_2 is the projection of z_2 onto $[p_1, p_3] \subseteq [p_1, z_3]$.

Continuing in this way, we construct sequences (p_i), (z_i) such that (p_i) is monotone increasing with respect to p_1, p_i is the projection of z_i onto $[p_1, p_{i+1}]$, $p_i \neq p_j$ for $i \neq j$, and $z_i \in \{y_n \mid n \geq 1\}$. Let $L = \bigcup_{i\geq 1}[p_1, p_i]$. Then L is a linear subtree with p_1 as an endpoint. By assumptions (i) and (ii), and criterion (iii) of Theorem 1, $L \subseteq [p_1, p]$ for some $p \in X$. Thus $[p_1, p] = [p_1, p_i, p_{i+1}, p_{i+2}, p]$, so $p_{i+1} \neq p$ and it follows that p_i is the projection of z_i onto $[p_1, p]$, for $i \geq 1$. But then $[p_1, p]$ contains infinitely many points p_i which are projections onto it of points in $\{y_n \mid n \geq 1\}$. This contradicts (3) and proves the theorem. □

We give two examples of complete, bounded, non-compact $\mathbb{R}$-trees.

Example 1. Let X_n be the set of points in $\mathbb{R}^2$ with polar coordinates $(r, \pi/n)$, where n is a positive integer and $0 \le r \le 1$. Then X_n is a copy of the unit interval $[0,1]$ with the distance between $(r, \pi/n)$ and $(r', \pi/n)$ equal to $|r - r'|$. By [2, Chapter 2, Lemma 1.13], $X := \bigcup_{n \ge 1} X_n$ is an $\mathbb{R}$-tree, where, for $m \ne n$, the distance from $(r, \pi/n)$ to $(r', \pi/m)$ is $r + r'$. Then X is not 1-locally finite at the origin 0, so (iii) fails, but (iv) holds as 0 is the only branch point. Also, (ii) holds as there are no open ends, and (i) clearly holds.

Example 2. Let (Y, d) be the $\mathbb{R}$-tree described before [2, Chapter 2, Lemma 2.7]. Thus $Y = \mathbb{R}^2$, and, using Cartesian coordinates,

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1| + |y_2| + |x_1 - x_2| & \text{if } x_1 \ne x_2 \\ |y_1 - y_2| & \text{otherwise.} \end{cases}$$

Then the degree of a point is at most four, so Y and any subtree satisfies (iii). The Euclidean unit square $[0,1] \times [0,1]$ is easily seen to be a subtree satisfying (i)–(iii). However, (iv) fails because the segment $[0,1]$ on the real axis does not have 1-locally finite branching. (Every point $x = (x, 0)$ of this segment is the projection of $(x, 1)$ onto it.)

We can use our results to characterise locally compact $\mathbb{R}$-trees. Let X be an $\mathbb{R}$-tree and let $x \in X$. If $y, z \in X$ are points contained in the closed ball $\bar{B}(x, r)$ of radius r around x, where $r > 0$, then the segment $[y, z]$ joining them is also contained in this ball, because $[y, z] \subseteq [x, y] \cup [x, z]$. Thus $\bar{B}(x, r)$ is a subtree of X.

Corollary 1. *An $\mathbb{R}$-tree X is locally compact if and only if it satisfies Conditions* (iii) *and* (iv) *of Theorem* 2, *as well as*

(v) *for each $x \in X$, there exists $s > 0$, such that $\bar{B}(x, s)$ has no open ends.*

Proof. Suppose that X is locally compact. Then, for $x \in X$, there exists $s > 0$ such that $\bar{B}(x, s)$ is compact. This closed ball is then bounded and complete, so has no open ends by Theorem 1. If $r > 0$ and there are infinitely many directions in X at x of length greater than or equal to r, then there are infinitely many directions in $\bar{B}(x, s)$ at x of length greater than or equal to $\min\{r, s\}$, contradicting Theorem 2. Hence, X satisfies (iii) and (v).

Suppose that (iv) fails to hold. Then there are a segment $[x, y]$ in X, a real number $r > 0$, and a sequence (p_i) of distinct points of X, such that there exists $z_n \in X$ with $d(z_n, p_n) = d(z_n, [x, y]) \ge r$. Replacing (p_n) by a subsequence, we may assume it converges to a point $p \in [x, y]$. Choose $s > 0$ such that $\bar{B}(p, s)$ is compact. Then $\bar{B}(p, s) \cap [x, y] = [x', y']$ for some x', $y' \in X$ with $x' \ne y'$.

Let w_n be the point on $[z_n, p_n]$ at distance $\min\{r, s/2\}$ from p_n. Infinitely many of the points p_n satisfy $d(p_n, p) \le s/2$ and these are in $[x', y']$. For such p_n, $[w_n, p_n] \cap [x', y'] = \{p_n\}$, because $[w_n, p_n] \subseteq [z_n, p_n]$ and $[x', y'] \subseteq [x, y]$. Also, for such p_n,

$$d(w_n, p) = d(w_n, p_n) + d(p_n, p) \le s/2 + s/2 = s,$$

so $w_n \in \bar{B}(p, s)$. But then (iv) fails in $\bar{B}(p, s)$, contradicting Theorem 2. Hence, X satisfies (iv).

Conversely, assume that X satisfies (iii)–(v). Let $x \in X$ and choose $s > 0$ such that $\bar{B}(x, s)$ has no open ends. Then $\bar{B}(x, s)$ is bounded, so is complete by Theorem 1. Clearly, $\bar{B}(x, s)$ satisfies (iii) and (iv), hence is compact by Theorem 2. Thus X is locally compact. □

If (X, d) is a complete locally compact $\mathbb{R}$-tree, then any closed ball $\bar{B}(x, r)$ is compact. For, by Corollary 1, X satisfies (iii) and (iv), hence $\bar{B}(x, r)$ satisfies (iii) and (iv). Compactness then follows from Theorem 2. This observation is made by Duquesne and Winkel [3], but using a version of the Hopf-Rinow Theorem given by Gromov [4].

It follows that a complete locally compact $\mathbb{R}$-tree is separable, and they go on to deduce that such an $\mathbb{R}$-tree has at most countably many branch points [3, Lemma 3.1]. For given a base point x_0 and a countable dense subset $\{y_i \mid i \geq 1\}$, any branch point is contained in one of the segments $[x_0, y_i]$ by Lemma 3(i), and segments have only countably many branch points. From our point of view, this follows from (iv) in Corollary 1. We do not know if this result is true without the assumption that the $\mathbb{R}$-tree is complete.

We remark that, by Condition (iii) in Corollary 1, the set of directions at every point of a locally compact $\mathbb{R}$-tree is at most countable (this fact is probably well-known, but we do not know a reference for it).

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