Abh. Math. Semin. Univ. Hambg. (2009) 79: 193–227
DOI 10.1007/s12188-009-0022-3

On a new construction in group theory

Thomas W. Müller · Jan-Christoph Schlage-Puchta

Received: 20 November 2008 / Published online: 14 March 2009

Abstract This paper continues the investigation of the groups $\mathcal{RF}(G)$ introduced and studied in [I.M. Chiswell and T.W. Müller, A class of groups with canonical $\mathbb{R}$-tree action, Springer LNM, to appear]. Two new concepts, that of a *test function*, and that of a pair of *locally incompatible* (test) functions are introduced, and their theory is developed. As application, we obtain a number of new quantitative as well as structural results concerning $\mathcal{RF}(G)$ and its quotient $\mathcal{RF}(G)/E(G)$ modulo the subgroup $E(G)$ generated by the elliptic elements. Among other things, the cardinality of $\mathcal{RF}(G)$ is determined, and it is shown that both $\mathcal{RF}(G)$ and $\mathcal{RF}(G)/E(G)$ contain large free subgroups, and that their abelianizations both contain a large $\mathbb{Q}$-vector space as direct summand.

Keywords $\mathbb{R}$-trees · Groups acting on $\mathbb{R}$-trees · Length functions

Mathematics Subject Classification (2000) 20E08 · 20F65 · 20E06

1 Introduction

In recent joint work of I.M. Chiswell and the first-named author, a powerful new construction was introduced and studied, which associates to each (discrete) group G a group $\mathcal{RF}(G)$ together with a canonical $\mathbb{R}$-tree action $\mathcal{RF}(G) \to \mathrm{Iso}(\mathbf{X}_G)$; cf. [3]. To some extent, in particular when working with their hyperbolic elements, these groups $\mathcal{RF}(G)$ appear as continuous analogues of free groups, whereas in other respects they behave more like amalgamated products, while in fact being neither. For the benefit of the reader, and since [3] has not yet appeared in print, Sect. 2 provides a quick introduction to some basic aspects of

Communicated by R. Diestel.

T.W. Müller
School of Mathematical Sciences, Queen Mary & Westfield College, University of London,
Mile End Road, E1 4NS London, UK

J.-C. Schlage-Puchta
Mathematisches Institut, Albert-Ludwigs-Universität, Eckerstr. 1, 79104 Freiburg, Germany

the theory developed in [3], as far as this is needed in the present context: we briefly review the definition of the group $\mathcal{RF}(G)$ itself, as well as that of its associated $\mathbb{R}$-tree $\mathbf{X}_G$, we comment in some detail on the action of $\mathcal{RF}(G)$ on $\mathbf{X}_G$, and describe the structure of the centralizer of an arbitrary hyperbolic element.

The purpose of the present paper is to explain some marked progress in our understanding of the $\mathcal{RF}$-construction, largely due to the introduction and exploitation of two basic new concepts: that of a *test function*, and that of a pair of *locally incompatible* test functions; see the beginnings of Sects. 3 and 5 for the relevant definitions. For instance, it was shown in [3] that two groups G and H having the same number of involutions as well as the same number of non-involutions satisfy $|\mathcal{RF}(G)| = |\mathcal{RF}(H)|$; but the actual cardinality of $\mathcal{RF}(G)$ could not be determined there. In Sect. 6, by constructing a large family of pairwise locally incompatible test functions, we show that in fact

$$|\mathcal{RF}(G)| = |G|^{2^{\aleph_0}}. \tag{1}$$

Further, it follows from the mere existence of test functions that $\mathcal{RF}(G)$ is never generated by its elliptic elements, a problem left open in [3] in the case when G is an elementary abelian 2-group. Also, Theorem 37 in Sect. 8 marks the beginning of a structure theory for $\mathcal{RF}(G)$ and its quotient modulo the subgroup $E(G)$ generated by the elliptic elements. Among other things, we show that (i) both $\mathcal{RF}(G)$ and $\mathcal{RF}(G)/E(G)$ contain a free subgroup of rank $|G|^{2^{\aleph_0}}$, but are not free; (ii) that every non-trivial torsion-free abelian group of rank at most $2^{\aleph_0}$ is realized (up to isomorphism) as centralizer of a hyperbolic element in $\mathcal{RF}(G)$; (iii) that the abelianized groups $\mathcal{RF}(G)/[\mathcal{RF}(G), \mathcal{RF}(G)]$ and $\mathcal{RF}(G)/E(G)[\mathcal{RF}(G), \mathcal{RF}(G)]$ both contain a $\mathbb{Q}$-vector space of dimension $|G|^{2^{\aleph_0}}$; and that (iv) every non-trivial normal subgroup $\mathcal{N}$ of $\mathcal{RF}(G)$ contains a free subgroup of rank $|G|^{2^{\aleph_0}}$; in particular, $|\mathcal{N}| = |\mathcal{RF}(G)|$.

We now describe the contents of the paper in more detail. After providing some necessary background material in Sect. 2, the concept of a test function is introduced in Sect. 3. We show there that test functions are cyclically reduced, and that non-trivial powers of test functions are again test functions. Also, by way of settling the existence problem, we exhibit a concrete example of a test function.

One of the principal problems of $\mathcal{RF}$-theory in its present state is the grave lack of (known) homomorphisms involving $\mathcal{RF}(G)$, a phenomenon largely due no doubt to the fact that no useful generating system (not to mention defining relations) is known for $\mathcal{RF}(G)$. In Sect. 4, we show how to associate with a given test function $f \in \mathcal{RF}(G)$ a certain surjective homomorphism $\lambda_f : \mathcal{RF}(G) \to \mathbb{R}$ via Lebesgue measure theory such that $\lambda_f(E(G)) = 0$. It follows in particular from this construction that $\mathcal{RF}(G)$ is never generated by its elliptic elements; in fact, it is shown in Sect. 8 that

$$(\mathcal{RF}(G) : E(G)) = |\mathcal{RF}(G)|.$$

Section 5 introduces the concept of a pair of locally incompatible functions. We prove that two locally incompatible functions in $\mathcal{RF}(G)$ exhibit no cancellation under multiplication, and we show how to build new test functions from old ones; cf. Proposition 29.

Section 6 contains the technical main result of our paper, viz. Theorem 30. Given a proper subgroup Λ of the additive reals, this result establishes existence of a family $\mathfrak{F}$ of pairwise locally incompatible test functions of size $|\mathfrak{F}| = |G|^{(\mathbb{R}:\Lambda)}$, such that $C_{\mathcal{RF}(G)}(f) \cong \Lambda$ for all $f \in \mathfrak{F}$. Choosing any Λ with $|\Lambda| = \aleph_0$, (1) follows easily, since test functions are reduced. We also obtain the assertion concerning centralizers in $\mathcal{RF}(G)$ mentioned before.

An alternative proof of (1) is provided in Sect. 7, this time relying on some of the rather paradoxical properties of the Cantor discontinuum; and the paper concludes with Sect. 8, which focuses on certain structural properties of $\mathcal{RF}(G)$ and $\mathcal{RF}(G)/E(G)$, most of which have already been mentioned.

2 The groups $\mathcal{RF}(G)$ and their associated $\mathbb{R}$-trees

2.1 Definition of the group $\mathcal{RF}(G)$

Given a group G, let $\mathcal{F}(G)$ be the set of all functions $f:[0,\alpha]\to G$ defined on some closed real interval $[0,\alpha]$ with $\alpha\geq 0$. The real number α will be called the *length* of the function f, denoted $L(f)$. The *formal inverse* f^{-1} of an element $f\in\mathcal{F}(G)$ is the function defined on the same interval $[0,\alpha]$ as f via

$$f^{-1}(\xi)=\big(f(\alpha-\xi)\big)^{-1},\quad 0\leq\xi\leq\alpha.$$

We have $(f^{-1})^{-1}=f$. A function $f\in\mathcal{F}(G)$ is *reduced*, if to every interior point ξ_0 in the domain of f with $f(\xi_0)=1_G$ and every real number ε satisfying $0<\varepsilon\leq\min\{\alpha-\xi_0,\xi_0\}$ there exists δ such that $0<\delta\leq\varepsilon$ and $f(\xi_0+\delta)\neq\big(f(\xi_0-\delta)\big)^{-1}$. Clearly, every element in $\mathcal{F}(G)$ of length 0 is reduced; and if f is reduced, then so is its formal inverse f^{-1}. We denote by $\mathcal{RF}(G)$ the set of all reduced functions in $\mathcal{F}(G)$.

We now proceed to define a multiplication on $\mathcal{F}(G)$ with the property that the product of two reduced functions is again reduced. Given $f,g\in\mathcal{F}(G)$ of lengths α respectively β, let

$$\varepsilon_0=\varepsilon_0(f,g):=\begin{cases}\sup\mathcal{E}(f,g), & f(\alpha)=\big(g(0)\big)^{-1},\\ 0, & \text{otherwise,}\end{cases}$$

where

$$\mathcal{E}(f,g):=\Big\{\varepsilon\in[0,\min\{\alpha,\beta\}]:\ f(\alpha-\delta)=\big(g(\delta)\big)^{-1}\text{ for all }\delta\in[0,\varepsilon]\Big\},$$

and define fg on the interval $[0,\alpha+\beta-2\varepsilon_0]$ via

$$(fg)(\xi):=\begin{cases}f(\xi), & 0\leq\xi<\alpha-\varepsilon_0\\ f(\alpha-\varepsilon_0)g(\varepsilon_0), & \xi=\alpha-\varepsilon_0\\ g(\xi-\alpha+2\varepsilon_0), & \alpha-\varepsilon_0<\xi\leq\alpha+\beta-2\varepsilon_0.\end{cases}$$

For later use, we note that

$$\varepsilon_0(f,g)=\varepsilon_0(g^{-1},f^{-1}),\quad f,g\in\mathcal{F}(G),$$

as the reader can verify without difficulty. We claim that the product fg of two reduced functions f and g is again reduced. This is clear if $f(\alpha-\varepsilon_0)g(\varepsilon_0)\neq 1_G$. So suppose that $f(\alpha-\varepsilon_0)g(\varepsilon_0)=1_G$, i.e., $\varepsilon_0\in\mathcal{E}(f,g)$, and that there exists ε' such that $0<\varepsilon'\leq\min\{\alpha-\varepsilon_0,\beta-\varepsilon_0\}$ and

$$(fg)(\alpha-\varepsilon_0-\delta)=\big((fg)(\alpha-\varepsilon_0+\delta)\big)^{-1},\quad 0<\delta\leq\varepsilon'.$$

By definition of fg this implies that

$$f(\alpha-\eta)=\big(g(\eta)\big)^{-1},\quad \varepsilon_0<\eta\le\varepsilon_0+\varepsilon', \tag{2}$$

while the fact that $\varepsilon_0\in\mathcal{E}(f,g)$ gives

$$f(\alpha-\eta)=\big(g(\eta)\big)^{-1},\quad 0\le\eta\le\varepsilon_0. \tag{3}$$

Combining Assertions (2) and (3), we conclude that $\varepsilon_0+\varepsilon'\in\mathcal{E}(f,g)$, implying $\varepsilon'\le 0$, a contradiction. Hence, fg is reduced as claimed.

Denote by $\mathbf{1}_G$ the function of length 0 with $\mathbf{1}_G(0)=1_G$. It is easy to see that, for $f\in\mathcal{F}(G)$,

$$\mathbf{1}_G f=f=f\mathbf{1}_G$$

and

$$ff^{-1}=\mathbf{1}_G=f^{-1}f,$$

which shows in particular that $\mathbf{1}_G$ is a neutral element for $\mathcal{RF}(G)$ with the above multiplication, and that the formal inverse f^{-1} of an element $f\in\mathcal{RF}(G)$ is its inverse. Moreover, one can show that our multiplication is associative on $\mathcal{RF}(G)$ (but not on $\mathcal{F}(G)$), although the proof of this fact as given in [3, Chap. 1] is quite non-trivial; hence, $\mathcal{RF}(G)$ when equipped with the multiplication defined above is in fact a group. We note that the group G we started from is embedded into $\mathcal{RF}(G)$ as the subgroup

$$G_0=\big\{f\in\mathcal{RF}(G):\ L(f)=0\big\}.$$

2.2 The star and circle products

There is another product on $\mathcal{F}(G)$, which is often useful in that its definition is more straightforward than that of reduced multiplication (and consequently computations run much easier than for the latter product), while the two products are nevertheless related in an important special case. For $f,g\in\mathcal{F}(G)$ of lengths α respectively β, define their star product $f*g$ as the function of length $\alpha+\beta$ satisfying

$$(f*g)(\xi)=\left\{\begin{array}{ll} f(\xi), & 0\le\xi<\alpha\\ f(\alpha)g(0), & \xi=\alpha\\ g(\xi-\alpha), & \alpha<\xi\le\alpha+\beta\end{array}\right\}\quad(\xi\in[0,\alpha+\beta]).$$

This multiplication is clearly analogous to concatenation of paths in topology. Straightforward computations show that the star product is associative and satisfies the cancellation rules

$$f*f_1=f*f_2\Longrightarrow f_1=f_2\quad\text{and}\quad f_1*f=f_2*f\Longrightarrow f_1=f_2\quad(f,f_1,f_2\in\mathcal{F}(G))$$

as well as

$$fg=f*g\quad(f,g\in\mathcal{F}(G),\ \varepsilon_0(f,g)=0).$$

In particular, we have

$$f*\mathbf{1}_G=f=\mathbf{1}_G*f,\quad f\in\mathcal{F}(G),$$

so that $(\mathcal{F}(G), *)$ is a cancellative semigroup with identity element ([3, Chap. 1, Prop. 1.1]). The following result computing values of long star products will be used later; cf. [3, Chap. 1, Lemma 1.2].

Lemma 1 *For $k \geq 1$, let $f_1, f_2, \dots, f_{k+1} \in \mathcal{F}(G)$ be functions such that $L(f_j) > 0$ for $2 \leq j \leq k$. For $1 \leq j \leq k+1$, set $\xi_j := \sum_{1 \leq i \leq j} L(f_i)$. Then we have*

$$(f_1 * f_2 * \cdots * f_{k+1})(\xi) = \left\{\begin{array}{ll} f_1(\xi), & 0 \leq \xi < \xi_1 \\ f_j(\xi - \xi_{j-1}), & \xi_{j-1} < \xi < \xi_j \; (2 \leq j \leq k) \\ f_{k+1}(\xi - \xi_k), & \xi_k < \xi \leq \xi_{k+1} \\ f_j(L(f_j)) f_{j+1}(0), & \xi = \xi_j \; (1 \leq j \leq k) \end{array}\right\}$$

$$(\xi \in [0, \xi_{k+1}]). \tag{4}$$

For $g_1, g_2 \in \mathcal{F}(G)$, we write $g_1 \circ g_2$ to mean $g_1 * g_2$ together with the information that $\varepsilon_0(g_1, g_2) = 0$, so that

$$g_1 \circ g_2 = g_1 * g_2 = g_1 g_2, \tag{5}$$

whenever $g_1 \circ g_2$ is defined; cf. Lemma 1.6 in [3, Chap. 1] and the remark following its proof. One can think of the $\circ$-operation as a partial multiplication on $\mathcal{F}(G)$, $g_1 \circ g_2$ being defined if, and only if, $\varepsilon_0(g_1, g_2) = 0$, in which case it equals $g_1 * g_2$. It can be shown that, if one of the products $(g_1 \circ g_2) \circ g_3$ and $g_1 \circ (g_2 \circ g_3)$ is defined, then so is the other ([3, Chap. 1, Cor. 1.14]); and the two product are then equal, since $*$-multiplication is associative.

2.3 The $\mathbb{R}$-tree associated with $\mathcal{RF}(G)$

It is not hard to see that the map $L : \mathcal{RF}(G) \to \mathbb{R}$ associating with each reduced function f the length $L(f)$ of its domain is a Lyndon length function.[1] As is well known, this yields (and is in fact equivalent to) the existence of an $\mathbb{R}$-tree $\mathbf{X}_G = (X_G, d_G)$ on which $\mathcal{RF}(G)$ acts, with a canonical base point x_0, and such that $L = L_{x_0}$, where

$$L_{x_0}(f) := d_G(x_0, f x_0), \quad f \in \mathcal{RF}(G)$$

is the displacement function associated with the action of $\mathcal{RF}(G)$ on $(\mathbf{X}_G, x_0)$. In particular, the stabilizer $\mathrm{stab}_{\mathcal{RF}(G)}(x_0)$ of the point x_0 under the action of $\mathcal{RF}(G)$ is given by

$$\mathrm{stab}_{\mathcal{RF}(G)}(x_0) = G_0.$$

In order to give the reader some feeling for this correspondence between real Lyndon length functions and $\mathbb{R}$-tree actions, we briefly describe the construction of $\mathbf{X}_G$ as pointed metric space.[2] Introduce an equivalence relation $\approx$ on $\mathcal{RF}(G)$ via

$$f \approx g :\Longleftrightarrow L(f^{-1}g) = 0,$$

[1] See, for instance, [2, p. 73] for the definition.

[2] See Theorems 4.4 and 4.6 in [2, Chap. 2] for more details.

and denote by $\langle f \rangle$ the equivalence class of $f \in \mathcal{RF}(G)$. One easily sees that

$$f \approx g \Longleftrightarrow L(f) = L(g) \quad \text{and}$$

$$f|_{[0,L(f))} = g|_{[0,L(g))} \Longleftrightarrow fG_0 = gG_0,$$

so that $\mathcal{RF}(G)/\approx$ is nothing but the coset space $\mathcal{RF}(G)/G_0$. Next, we form the set

$$Y_G = \Big\{(\langle f \rangle, \alpha) : \, f \in \mathcal{RF}(G), \, \alpha \in \mathbb{R}, \, 0 \le \alpha \le L(f)\Big\},$$

and introduce an equivalence relation $\sim$ on Y_G via

$$(\langle f \rangle, \alpha) \sim (\langle g \rangle, \beta) :\Longleftrightarrow \varepsilon_0(f^{-1}, g) \ge \alpha = \beta.$$

We denote the equivalence class of $(\langle f \rangle, \alpha)$ by $\langle f, \alpha \rangle$, observing that we always have $\langle f, \alpha \rangle = \langle f|_{[0,\alpha]}, \alpha \rangle$. Then

$$X_G = Y_G/\sim,$$

$$d_G(\langle f, \alpha \rangle, \langle g, \beta \rangle) = \alpha + \beta - 2\min\big\{\alpha, \beta, \varepsilon_0(f^{-1}, g)\big\},$$

and

$$x_0 = \langle \mathbf{1}_G, 0 \rangle.$$

It can be shown that $\mathbf{X}_G$ is metrically complete; cf. [3, Chap. 2, Prop. 2.4].

2.4 The action of $\mathcal{RF}(G)$ on $\mathbf{X}_G$

Whenever a group Γ acts on an $\mathbb{R}$-tree $\mathbf{X}$, we can classify the elements of Γ according to whether they are *elliptic* (i.e., possess a fixed point) or *hyperbolic* (i.e., act as a fixed-point-free isometry on $\mathbf{X}$). Hyperbolic elements have some local geometry associated to them: if $\gamma \in \Gamma$ is hyperbolic, then there exists an isometric copy $A_\gamma \subseteq \mathbf{X}$ of the real line (the so-called *axis* of γ) such that γ acts on A_γ as a non-trivial translation; in particular, hyperbolic elements have infinite order. The translation length of a hyperbolic element γ on its axis A_γ is called the *hyperbolic length* of γ, denoted $\ell(\gamma)$; and one defines $\ell(\gamma)$ to be zero, if $\gamma \in \Gamma$ is elliptic. It is shown in [3] that the action of $\mathcal{RF}(G)$ on the $\mathbb{R}$-tree $\mathbf{X}_G$ is in fact *transitive*; cf. [3, Sect. 2.4]. It follows that the set of elliptic elements of $\mathcal{RF}(G)$ equals

$$\bigcup_{t \in \mathcal{RF}(G)} tG_0t^{-1},$$

the union of all the conjugates of the subgroup G_0 of length zero functions in $\mathcal{RF}(G)$. As a consequence, we see that the group $\mathcal{RF}(G)$ is torsion-free if, and only if, G is torsion-free. In fact, more is true.

Proposition 2 *Let $\mathcal{H}$ be a subgroup of $\mathcal{RF}(G)$. Then the following assertions are equivalent.*

(i) *$\mathcal{H}$ is bounded.*
(ii) *$\mathcal{H}$ consist entirely of elliptic elements.*
(iii) *$\mathcal{H}$ is conjugate to a subgroup of G_0.*

This is [3, Chap. 2, Prop. 2.24]. Moreover, it is not hard to see that, for a non-trivial subgroup U of G_0, we have

$$N_{\mathcal{RF}(G)}(U) = N_{G_0}(U); \tag{6}$$

in particular, G_0 is self-normalizing in $\mathcal{RF}(G)$; cf. [3, Chap. 1, Prop. 1.16(ii)]. As a consequence of Proposition 2 and (6), we have the following important observation.

Corollary 3 *The only bounded normal subgroup in $\mathcal{RF}(G)$ is the trivial group $\{\mathbf{1}_G\}$.*

Proof The assertion is trivial if $G = \{1_G\}$, so we may assume that G is non-trivial, and consequently, $G_0 < \mathcal{RF}(G)$. Suppose that $\mathcal{N} \trianglelefteq \mathcal{RF}(G)$ is a non-trivial bounded normal subgroup. Then $\mathcal{N} \leq G_0$ by Proposition 2 plus normality of $\mathcal{N}$, hence by (6),

$$\mathcal{RF}(G) = N_{\mathcal{RF}(G)}(\mathcal{N}) = N_{G_0}(\mathcal{N}) = G_0,$$

a contradiction. □

In order to be able to better describe the action of $\mathcal{RF}(G)$ on $\mathbf{X}_G$, we need to introduce the concept of a *cyclically reduced* function, and to explain the process of *cyclic reduction*.

Definition 4 A function $f \in \mathcal{RF}(G)$ is called *cyclically reduced*, if $\varepsilon_0(f, f) = 0$; or, equivalently, if $L(f^2) = 2L(f)$.

Clearly, every function of length 0 is cyclically reduced; and, if $f \in \mathcal{RF}(G)$ is cyclically reduced, then so is every power f^k with $k \in \mathbb{Z}$; cf. Part (ii) of Lemma 2.6 in [3, Chap. 2]. The following important result, which is [3, Chap. 2, Lemma 2.7], describes existence and uniqueness of the cyclically reduced *core* of a reduced function f.

Proposition 5 (i) *Let $f \in \mathcal{RF}(G)$. Then there exist $t, f_1 \in \mathcal{RF}(G)$, such that $f = t \circ f_1 \circ t^{-1}$, and f_1 is cyclically reduced.*

(ii) *If $f = t \circ f_1 \circ t^{-1} = s \circ f_2 \circ s^{-1}$, where $t, s, f_1, f_2 \in \mathcal{RF}(G)$, and f_1, f_2 are cyclically reduced, then $s = tg$ and $f_2 = g^{-1} f_1 g$ for some $g \in G_0$.*

Definition 6 The function f_1 described for given $f \in \mathcal{RF}(G)$ in Proposition 5, which is unique up to conjugation by a G_0-element, is called the (cyclically reduced) *core* of f, denoted $c(f)$. The passage from f to f_1 is called *cyclic reduction* of f.

The importance of cyclic reduction in the present context stems from the fact that it allows us to characterise in algebraic terms when a reduced function f is hyperbolic; it also enables us to compute the hyperbolic length of f; cf. Proposition 2.16(b) and Corollary 2.21 in [3, Chap. 2].

Proposition 7 *A function $f \in \mathcal{RF}(G)$ is hyperbolic if, and only if, its core $c(f)$ is a function of positive length, i.e., $L(c(f)) > 0$. Moreover, we have*

$$\ell(f) = L(c(f)), \quad f \in \mathcal{RF}(G).$$

Another useful consequence of Proposition 5 is the fact that the conjugates of G_0 in $\mathcal{RF}(G)$ form an amalgam with trivial intersection.

Corollary 8 *Let sG_0s^{-1} and tG_0t^{-1} be any two distinct conjugates of G_0 in $\mathcal{RF}(G)$. Then $sG_0s^{-1} \cap tG_0t^{-1} = \{\mathbf{1}_G\}$.*

Proof Suppose that $sG_0s^{-1} \cap tG_0t^{-1} \neq \{\mathbf{1}_G\}$, and let x be a non-trivial element in this intersection. Then

$$s \circ g \circ s^{-1} = x = t \circ h \circ t^{-1}$$

for some $g, h \in G_0$. Since elements of length 0 are cyclically reduced, Part(ii) of Proposition 5 gives $s = tk$ for some $k \in G_0$, hence $sG_0s^{-1} = tG_0t^{-1}$. □

2.5 Centralizers

Since an element of length 0 cannot commute with a function of positive length, we have

$$C_{\mathcal{RF}(G)}(g) = C_{G_0}(g), \quad g \in G_0 \setminus \{\mathbf{1}_G\}.$$

Consequently, the centralizers in $\mathcal{RF}(G)$ of elliptic elements are determined up to isomorphism through the centralizer structure of G itself; and nothing further can be said here in general. The situation is entirely different for hyperbolic elements. In order to be able to state a precise result, we first need to explain the concept of a *(strong) period* of a reduced function f.

Definition 9 Let $f \in \mathcal{RF}(G)$ be an element of length $L(f) = \alpha > 0$.

(i) The points $\omega \in [0, \alpha]$ satisfying

$$\forall \gamma, \delta \in (0, \alpha] : |\gamma - \delta| = \omega \to f(\gamma) = f(\delta)$$

are called *periods* of f. The set of all periods of f is denoted by Ω_f.

(ii) The elements of the set

$$\Omega_f^0 = \{\omega \in \Omega_f : \alpha - \omega \in \Omega_f\}$$

are termed *strong periods* of f.

By Part (i) of Proposition 5 together with Proposition 7, every hyperbolic function $f \in \mathcal{RF}(G)$ is conjugate to a cyclically reduced function f_1 of positive length, which is *normalized* in the sense that $f_1(0) = 1_G$. Our next result, which is part of [3, Chap. 6, Theorem 6.13], describes the centralizer in $\mathcal{RF}(G)$ of such a function f_1 as a *subset* of $\mathcal{RF}(G)$, while also providing an *isomorphic model* for $C_{\mathcal{RF}(G)}(f_1)$ via a subgroup of $(\mathbb{R}, +)$.

Theorem 10 *Let $f \in \mathcal{RF}(G)$ be cyclically reduced, of length $L(f) = \alpha > 0$, and normalized. Then the set*

$$C_f := \left\{ f^k \circ f|_{[0,\omega]} : (k, \omega) \in \mathbb{N}_0 \times \left(\Omega_f^0 \setminus \{\alpha\}\right), \ k + \omega > 0 \right\}$$

forms a positive cone for the centralizer $C_{\mathcal{RF}(G)}(f)$ of f in $\mathcal{RF}(G)$, giving $C_{\mathcal{RF}(G)}(f)$ the structure of an ordered abelian group. Moreover, the mapping $\rho_f : C_{\mathcal{RF}(G)}(f) \to \langle \Omega_f^0 \rangle$ given by

$$\left(f^k \circ f|_{[0,\omega]}\right)^\sigma \mapsto \sigma(k\alpha + \omega), \quad \sigma \in \{-1, 1\},$$

is an isomorphism of ordered abelian groups.

3 Test functions

Definition 11 A function $f \in \mathcal{F}(G)$ is called a *test function*, if $L(f) > 0$, and there do not exist $\varepsilon > 0$ and points $\xi_1, \xi_2 \in (0, L(f))$ such that

$$f(\xi_1 + \eta) = f^{-1}(\xi_2 + \eta), \quad |\eta| < \varepsilon.$$

Clearly, the inverse of a test function is again a test function. A first important observation is that test functions are automatically reduced. In fact, we can prove slightly more.

Lemma 12 *Test functions are cyclically reduced.*

Proof Let f be a test function. We first show that f is reduced. Suppose by way of contradiction that there exists an interior point ξ_0 of the domain $[0, L(f)]$ such that $f(\xi_0) = 1_G$, and a cancelling ε-neighbourhood for f around ξ_0. By definition, this means that, for $|\eta| < \varepsilon$,

$$f(\xi_0 + \eta) = \big(f(\xi_0 - \eta)\big)^{-1} = f^{-1}(L(f) - \xi_0 + \eta) = f^{-1}(\xi_0' + \eta),$$

where $\xi_0' := L(f) - \xi_0$ is again an inner point of the domain of f. Since the resulting equation

$$f(\xi_0 + \eta) = f^{-1}(\xi_0' + \eta), \quad |\eta| < \varepsilon$$

contradicts the definition of a test function, f is indeed reduced, as claimed.

Next, suppose that f is not cyclically reduced; that is, that $\varepsilon_0(f, f) > 0$. Then there exists $\varepsilon > 0$ such that

$$f(\alpha - \eta) f(\eta) = 1_G, \quad 0 \le \eta \le \varepsilon.$$

Rewriting the last equation as $f(\eta) = f^{-1}(\eta)$, and setting $\eta = \frac{\varepsilon}{2} + \eta'$, we find that

$$f\left(\frac{\varepsilon}{2} + \eta'\right) = f^{-1}\left(\frac{\varepsilon}{2} + \eta'\right), \quad |\eta'| < \frac{\varepsilon}{2},$$

again contradicting the definition of a test function. □

For later usage, we also record the following.

Lemma 13 *Let f be a test function of length $L(f) = \alpha$, let k be a non-negative integer, and let α' be a real number such that $0 \le \alpha' \le \alpha$. Then the function $g = f^k f|_{[0,\alpha']}$ is again a test function, provided that $k + \alpha' > 0$.*

Proof By Lemma 12, f is cyclically reduced, so that

$$g = \underbrace{f * \cdots * f}_{k \text{ times}} * f|_{[0,\alpha']};$$

in particular, $L(g) = k\alpha + \alpha'$. Assume for a contradiction that there exists $\varepsilon > 0$ and points $\xi_1, \xi_2 \in (0, L(g))$, such that

$$g(\xi_1 + \eta) = g^{-1}(\xi_2 + \eta), \quad |\eta| < \varepsilon. \tag{7}$$

We now distinguish three cases.

(i) $k=0$. Then $g=f|_{[0,\alpha']}$ with $\alpha'>0$; and, for $|\eta|<\varepsilon$ and ε sufficiently small, we have

$$g(\xi_1+\eta)=f(\xi_1+\eta),$$

while

$$\begin{aligned}g^{-1}(\xi_2+\eta)&=\big(f|_{[0,\alpha']}\big)^{-1}(\xi_2+\eta)\\&=\big(f(\alpha'-\xi_2-\eta)\big)^{-1}\\&=f^{-1}(\alpha-\alpha'+\xi_2+\eta).\end{aligned}$$

It follows from (7) that

$$f(\xi_1+\eta)=f^{-1}(\alpha-\alpha'+\xi_2+\eta),\quad |\eta|<\varepsilon,$$

contradicting the fact that f itself is a test function, since we have

$$0<\xi_1,\alpha-\alpha'+\xi_2<\alpha.$$

(ii) $\alpha'=0$. Now we have $g=f^k$ with $k\geq 1$. By Lemma 1 with k replaced by $k-1$ and $f_j=f$ for $j=1,2,\dots,k$, we find that the values of g may be computed via the formula

$$g(\xi)=\left\{\begin{array}{ll}f(\xi \bmod \alpha), & \xi\notin\{\alpha,2\alpha,\dots,k\alpha\}\\ f(\alpha)f(0), & \xi\in\{\alpha,2\alpha,\dots,(k-1)\alpha\}\\ f(\alpha), & \xi=k\alpha\end{array}\right\}\quad (\xi\in[0,k\alpha]). \tag{8}$$

By moving ξ_1 and ξ_2 slightly if necessary, and decreasing ε accordingly, we may suppose that

$$\xi_1,\xi_2\notin\big\{\alpha,2\alpha,\dots,(k-1)\alpha\big\}.$$

Let

$$(i-1)\alpha<\xi_1<i\alpha$$

and

$$(j-1)\alpha<\xi_2<j\alpha$$

for some $1\leq i,j\leq k$. Then, according to (8), for $|\eta|<\varepsilon$ and sufficiently small ε,

$$g(\xi_1+\eta)=f(\xi_1-(i-1)\alpha+\eta),$$

while

$$\begin{aligned}g^{-1}(\xi_2+\eta)&=\big(g(k\alpha-\xi_2-\eta)\big)^{-1}\\&=\big(f(j\alpha-\xi_2-\eta)\big)^{-1}\\&=f^{-1}(\xi_2-(j-1)\alpha+\eta).\end{aligned}$$

Hence, (7) implies that

$$f(\xi_1 - (i-1)\alpha + \eta) = f^{-1}(\xi_2 - (j-1)\alpha + \eta), \quad |\eta| < \varepsilon,$$

again contradicting the fact that f itself is a test function, since

$$0 < \xi_1 - (i-1)\alpha,\ \xi_2 - (j-1)\alpha < \alpha.$$

(iii) $k > 0$ *and* $0 < \alpha' < \alpha$. Using Lemma 1 again, this time with $f_j = f$ for $j = 1, 2, \ldots, k$ and $f_{k+1} = f|_{[0,\alpha']}$, we find that g-values may be computed via

$$g(\xi) = \begin{Bmatrix} f(\xi \bmod \alpha), & \xi \notin \{\alpha, 2\alpha, \ldots, k\alpha\} \\ f(\alpha)f(0), & \xi \in \{\alpha, 2\alpha, \ldots, k\alpha\} \end{Bmatrix} \quad (\xi \in [0, k\alpha + \alpha']),$$

and the rest of the argument proceeds in a manner analogous to that of Case (ii). □

Corollary 14 *If f is a test function and $k \in \mathbb{Z} \setminus \{0\}$, then f^k is again a test function.*

Proof Since the inverse of a test function is again a test function, we may suppose that k is a positive integer, in which case the result follows from Lemma 13 with $\alpha' = 0$. □

Test functions do in fact exist; this will follow from a much stronger result, demonstrating the existence of large families of 'mutually independent' test functions with prescribed centralizer; cf. Theorem 30 in Sect. 6. For the moment, we confine ourselves with exhibiting just one concrete example. Let $x \in G$ be a non-trivial element. Then the function f_0 of length 1 given via

$$f_0(\xi) = \begin{Bmatrix} x, & \xi^2 \in \mathbb{Q} \\ 1_G, & \xi^2 \notin \mathbb{Q} \end{Bmatrix} \quad (0 \le \xi \le 1)$$

is a test function. Indeed, suppose that there exist $\varepsilon > 0$ and points $\xi_1, \xi_2 \in (0, 1)$ such that

$$f_0(\xi_1 + \eta) = f_0^{-1}(\xi_2 + \eta), \quad |\eta| < \varepsilon. \tag{9}$$

Choosing η in (9) such that $\xi_1 + \eta$ is rational, we see that (9) is impossible if $x^2 \neq 1_G$; thus, we may suppose that $x = x^{-1}$ is a non-trivial involution, so that (9) simplifies to

$$f_0(\xi_1 + \eta) = f_0(\xi_2' - \eta), \quad |\eta| < \varepsilon, \tag{10}$$

where $\xi_2' := 1 - \xi_2$. Equation (10) in turn is equivalent to the assertion that

$$(\xi_1 + \eta)^2 \in \mathbb{Q} \Longleftrightarrow (-\xi_2' + \eta)^2 \in \mathbb{Q}, \quad |\eta| < \varepsilon,$$

which is seen to be impossible; cf. Corollary 35 in Sect. 6.2.

4 The maps λ_f

4.1 The sets $\mathcal{M}_f^+(g)$ and $\mathcal{M}_f^-(g)$

Given a test function f of length α, and an arbitrary element $g:[0,\beta]\to G$ of $\mathcal{F}(G)$, we define sets $\mathcal{M}_f^+(g)$ and $\mathcal{M}_f^-(g)$ via

$$\mathcal{M}_f^+(g) := \Big\{\xi \in (0,\beta) : \exists\, \varepsilon > 0,\ \exists\, \xi' \in (0,\alpha) \text{ s.t. } g(\xi+\eta) = f(\xi'+\eta) \text{ for all } |\eta| < \varepsilon\Big\}$$

and

$$\mathcal{M}_f^-(g) := \Big\{\xi \in (0,\beta) : \exists\, \varepsilon > 0,\ \exists\, \xi' \in (0,\alpha) \text{ s.t. } g(\xi+\eta) = f^{-1}(\xi'+\eta) \text{ for all } |\eta| < \varepsilon\Big\}.$$

Of course, our notation is supposed to imply that all function values written down are actually defined; that is, that ε satisfies the inequality

$$\varepsilon \le \min\big\{\xi, \xi', \beta-\xi, \alpha-\xi'\big\}.$$

Our next observation is as follows.

Lemma 15 *Let f be any fixed test function. Then*

$$\mathcal{M}_f^+(g) \cap \mathcal{M}_f^-(g) = \emptyset, \quad g \in \mathcal{F}(G).$$

Proof Assume for a contradiction that $\xi_0 \in \mathcal{M}_f^+(g) \cap \mathcal{M}_f^-(g)$ for some $g \in \mathcal{F}(G)$, and let $L(f)=\alpha$, $L(g)=\beta$. Then $\xi_0 \in (0,\beta)$, and there exist $\varepsilon > 0$ as well as points $\xi_1, \xi_2 \in (0,\alpha)$ such that

$$\left\{\begin{aligned} g(\xi_0+\eta) &= f(\xi_1+\eta)\\ g(\xi_0+\eta) &= f^{-1}(\xi_2+\eta)\end{aligned}\right\} \quad (|\eta| < \epsilon). \tag{11}$$

We conclude from (11) that

$$f(\xi_1+\eta) = f^{-1}(\xi_2+\eta), \quad |\eta| < \varepsilon,$$

contradicting the fact that f is a test function. Hence, the sets $\mathcal{M}_f^+(g)$ and $\mathcal{M}_f^-(g)$ are indeed disjoint for every $g \in \mathcal{F}(G)$, as claimed. □

Corollary 16 *If f is a test function of length α, then, for $0 \le \alpha' \le \alpha$, we have*

$$\mathcal{M}_f^+(f|_{[0,\alpha']}) = (0,\alpha'), \tag{12}$$

and

$$\mathcal{M}_f^-(f|_{[0,\alpha']}) = \emptyset. \tag{13}$$

Proof Equation (12) is clear by definition, and (13) follows from (12) together with Lemma 15. □

We also note that, for each test function f,

$$\mathcal{M}_f^+(g) = \mathcal{M}_f^-(g) = \emptyset, \quad g \in G_0. \tag{14}$$

4.2 Definition of the maps λ_f

Since the sets $\mathcal{M}_f^+(g)$, $\mathcal{M}_f^-(g)$ are defined via open conditions, that is, conditions invariant under slight perturbation of the point considered, $\mathcal{M}_f^+(g)$ and $\mathcal{M}_f^-(g)$ themselves are open sets, thus Lebesgue measurable. Given a test function f, and defining the sets $\mathcal{M}_f^+(g)$ and $\mathcal{M}_f^-(g)$ for $g \in \mathcal{RF}(G)$ as described in the previous subsection, we introduce a function $\lambda_f : \mathcal{RF}(G) \to \mathbb{R}$ via

$$\lambda_f(g) := \mu\big(\mathcal{M}_f^+(g)\big) - \mu\big(\mathcal{M}_f^-(g)\big), \quad g \in \mathcal{RF}(G),$$

where μ denotes the Lebesgue measure. We observe that, by (14),

$$\lambda_f(G_0) = 0. \tag{15}$$

Moreover, by Corollary 16, we have

$$\lambda_f(f|_{[0,\beta]}) = \beta, \quad 0 \le \beta \le L(f) \tag{16}$$

(note that restrictions of f are reduced since f itself is reduced by Lemma 12).

4.3 λ_f respects inverses

Our next goal is to show that λ_f is in fact a homomorphism; in preparation for this argument, we first observe that λ_f respects inverses.

Lemma 17 *For each fixed test function f, we have*

$$\lambda_f(g^{-1}) = -\lambda_f(g), \quad g \in \mathcal{RF}(G). \tag{17}$$

Proof Suppose that $L(f) = \alpha$ and $L(g) = \beta$. Then, by definition,

$$\mathcal{M}_f^+(g^{-1}) = \Big\{\xi \in (0,\beta) : \exists\, \varepsilon > 0,\ \exists\, \xi_1 \in (0,\alpha) \text{ s.t. } g^{-1}(\xi+\eta) = f(\xi_1+\eta) \text{ for all } |\eta| < \varepsilon\Big\}.$$

Since, for $\xi \in \mathcal{M}_f^+(g)$ with corresponding ξ_1, ε and $|\eta| < \varepsilon$,

$$\big(g(\beta - \xi - \eta)\big)^{-1} = g^{-1}(\xi+\eta) = f(\xi_1+\eta) = \big(f^{-1}(\alpha - \xi_1 - \eta)\big)^{-1},$$

we find that, for $\xi \in (0,\beta)$,

$$\xi \in \mathcal{M}_f^+(g) \iff \exists\, \varepsilon > 0,\ \exists\, \xi_1' \in (0,\alpha) \text{ s.t. } g(\xi'+\eta') = f^{-1}(\xi_1'+\eta') \text{ for all } |\eta| < \varepsilon$$

$$\iff \xi' \in \mathcal{M}_f^-(g),$$

where $\xi' := \beta - \xi$, $\xi_1' := \alpha - \xi_1$, and $\eta' = -\eta$. We deduce that

$$\mathcal{M}_f^+(g^{-1}) = -\big(\mathcal{M}_f^-(g)\big) + \beta, \tag{18}$$

and replacing g with g^{-1} in (18) yields the corresponding formula

$$\mathcal{M}_f^-(g^{-1}) = -\big(\mathcal{M}_f^+(g)\big) + \beta. \tag{19}$$

The well-known behaviour of Lebesgue measure under linear transformations[3] together with Formulae (18) and (19) now implies that

$$\mu\big(\mathcal{M}_f^+(g^{-1})\big) = \mu\big(\mathcal{M}_f^-(g)\big)$$

and

$$\mu\big(\mathcal{M}_f^-(g^{-1})\big) = \mu\big(\mathcal{M}_f^+(g)\big),$$

from which we infer that

$$\begin{aligned}\lambda_f(g^{-1}) &= \mu\big(\mathcal{M}_f^+(g^{-1})\big) - \mu\big(\mathcal{M}_f^-(g^{-1})\big)\\ &= \mu\big(\mathcal{M}_f^-(g)\big) - \mu\big(\mathcal{M}_f^+(g)\big)\\ &= -\lambda_f(g);\end{aligned}$$

that is, Formula (17). □

4.4 Visibility of cancellation

The following auxiliary result, which is [3, Chap. 1, Lemma 1.11], plays a crucial role in the proof of our first main result (Theorem 19 below).

Lemma 18 *Let $g, h \in \mathcal{RF}(G)$ be reduced functions. Then there exist $g_1, h_1, c \in \mathcal{RF}(G)$ such that $g = g_1 \circ c$, $h = c^{-1} \circ h_1$, and $gh = g_1 \circ h_1$.*

Because of its importance in the present context, we briefly sketch the proof of Lemma 18. First, it is not hard to see that, given $g \in \mathcal{RF}(G)$ of length $L(g) = \beta$ and a real number γ such that $0 \le \gamma \le \beta$, there exist functions $g_1, g_2 \in \mathcal{RF}(G)$ such that $g = g_1 \circ g_2$ and $L(g_1) = \gamma$; moreover, one finds that, once one of the values $g_1(\gamma)$, $g_2(0)$ has been specified, g_1 and g_2 are in fact uniquely determined, and that one of these values may be chosen arbitrarily in G (see [3, Chap. 1, Lemma 1.10] for more details).

If $\varepsilon_0 := \varepsilon_0(g, h) = 0$, then the conclusion of Lemma 18 is satisfied with $g_1 := g$, $h_1 := h$, and $c := \mathbf{1}_G$; hence, we may suppose that $\varepsilon_0 > 0$. Decomposing g and h in accordance with the above remark as $g = g_1 \circ c$ respectively $h = d \circ h_1$ with $L(c) = L(d) = \varepsilon_0$, a straightforward argument using the fact that $\varepsilon_0 > 0$ shows that

$$\big(c(\varepsilon_0 - \eta)\big)^{-1} = d(\eta), \quad 0 \le \eta < \varepsilon_0;$$

that is, c^{-1} and d agree everywhere except possibly on their endpoints. Since $c(0)$ and $d(\varepsilon_0)$ can be chosen arbitrarily (and independently of each other), we can certainly arrange that

$$\big(c(0)\big)^{-1} = d(\varepsilon_0),$$

so that indeed $d = c^{-1}$. Finally, noting that

$$L(gh) = L(g) + L(h) - 2\varepsilon_0 = L(g_1) + L(h_1) = L(g_1 * h_1),$$

and comparing the values of the function gh with those of $g_1 * h_1$, we find after some calculation that $gh = g_1 \circ h_1$, as required.

[3] Cf., for instance, [6, Chap. III, § 15, Theorem D].

4.5 First main result

This is the following.

Theorem 19 *For each fixed test function f, the map $\lambda_f : \mathcal{RF}(G) \to \mathbb{R}$ defined in Sect. 4.2 is a surjective homomorphism of groups whose kernel contains $E(G)$, the subgroup generated by the elliptic elements of $\mathcal{RF}(G)$.*

Proof Suppose first that $g, h \in \mathcal{F}(G)$ and that $L(g) = \beta$. Then we claim that

$$\mathcal{M}_f^+(g * h) - \{\beta\} = \mathcal{M}_f^+(g) \cup \big(\mathcal{M}_f^+(h) + \beta\big) \tag{20}$$

and

$$\mathcal{M}_f^-(g * h) - \{\beta\} = \mathcal{M}_f^-(g) \cup \big(\mathcal{M}_f^-(h) + \beta\big). \tag{21}$$

Indeed, let $L(f) = \alpha$ and $L(h) = \gamma$. Then, for $\xi \in (0, \beta + \gamma)$,

$$\begin{aligned}
\xi \in \mathcal{M}_f^+(g * h) - \{\beta\} \iff & \Big(\exists\, \varepsilon > 0,\ \exists\, \xi' \in (0, \alpha) \text{ s.t. } (\xi - \varepsilon, \xi + \varepsilon) \subseteq (0, \beta) \text{ and} \\
& (g * h)(\xi + \eta) = f(\xi' + \eta) \text{ for all } |\eta| < \varepsilon\Big) \\
& \text{or} \\
& \Big(\exists\, \varepsilon > 0,\ \exists\, \xi' \in (0, \alpha) \text{ s.t. } (\xi - \varepsilon, \xi + \varepsilon) \subseteq (\beta, \beta + \gamma) \text{ and} \\
& (g * h)(\xi + \eta) = f(\xi' + \eta) \text{ for all } |\eta| < \varepsilon\Big) \\
\iff & \Big(\exists\, \varepsilon > 0,\ \exists\, \xi' \in (0, \alpha) \text{ s.t. } g(\xi + \eta) = f(\xi' + \eta) \text{ for all } |\eta| < \varepsilon\Big) \\
& \text{or} \\
& \Big(\exists\, \varepsilon > 0,\ \exists\, \xi' \in (0, \alpha) \text{ s.t. } h(\xi - \beta + \eta) = f(\xi' + \eta) \text{ for all } |\eta| < \varepsilon\Big) \\
\iff & \xi \in \mathcal{M}_f^+(g) \text{ or } \xi - \beta \in \mathcal{M}_f^+(h) \\
\iff & \xi \in \mathcal{M}_f^+(g) \cup \big(\mathcal{M}_f^+(h) + \beta\big),
\end{aligned}$$

whence (20). The proof of (21) is similar. Since a singleton set has measure 0, and Lebesgue measure is invariant under translations, we infer from (20) and (21) that

$$\mu\big(\mathcal{M}_f^+(g * h)\big) = \mu\big(\mathcal{M}_f^+(g)\big) + \mu\big(\mathcal{M}_f^+(h)\big) \tag{22}$$

and

$$\mu\big(\mathcal{M}_f^-(g * h)\big) = \mu\big(\mathcal{M}_f^-(g)\big) + \mu\big(\mathcal{M}_f^-(h)\big). \tag{23}$$

Combining Formulae (22) and (23) with (5), we now find that, for $g, h \in \mathcal{RF}(G)$ such that $\varepsilon_0(g, h) = 0$,

$$\begin{aligned}
\lambda_f(gh) &= \mu\big(\mathcal{M}_f^+(gh)\big) - \mu\big(\mathcal{M}_f^-(gh)\big) \\
&= \mu\big(\mathcal{M}_f^+(g * h)\big) - \mu\big(\mathcal{M}_f^-(g * h)\big)
\end{aligned}$$

$$= \Big(\mu\big(\mathcal{M}_f^+(g)\big) + \mu\big(\mathcal{M}_f^+(h)\big)\Big) - \Big(\mu\big(\mathcal{M}_f^-(g)\big) + \mu\big(\mathcal{M}_f^-(h)\big)\Big)$$

$$= \lambda_f(g) + \lambda_f(h);$$

that is, the equation

$$\lambda_f(gh) = \lambda_f(g) + \lambda_f(h) \quad (g, h \in \mathcal{RF}(G)) \tag{24}$$

has been verified whenever g and h are such that $\varepsilon_0(g, h) = 0$.

Now let $g, h \in \mathcal{RF}(G)$ be arbitrary, and apply Lemma 18 to write $g = g_1 \circ c$, $h = c^{-1} \circ h_1$ so that $gh = g_1 \circ h_1$. Then, using the last observation together with Lemma 17, we find that

$$\begin{aligned}\lambda_f(gh) &= \lambda_f(g_1 \circ h_1)\\ &= \lambda_f(g_1) + \lambda_f(h_1)\\ &= \lambda_f(g_1) + \lambda_f(c) + \lambda_f(c^{-1}) + \lambda_f(h_1)\\ &= \lambda_f(g_1 \circ c) + \lambda_f(c^{-1} \circ h_1)\\ &= \lambda_f(g) + \lambda_f(h),\end{aligned}$$

so that (24) holds in general; that is, λ_f is a group homomorphism. By (16), we have

$$[0, \alpha] \subseteq \lambda_f(\mathcal{RF}(G)),$$

which, since $\alpha > 0$, is more than enough to conclude that λ_f is surjective, and the last assertion of Theorem 19 follows from (15) together with the fact that $E(G) = \langle\langle G_0 \rangle\rangle$ is the normal closure of G_0. □

As an immediate consequence of Theorem 19 and the existence of test functions (see the end of Sect. 3), we have established the following.

Corollary 20 *Let G be a non-trivial group. Then the quotient group $\mathcal{RF}(G)/E(G)$ maps homomorphically onto $\mathbb{R}$; in particular, $\mathcal{RF}(G)$ is not generated by its elliptic elements.*

We also obtain the following.

Corollary 21 *If $f \in \mathcal{RF}(G)$ is a test function, then $f \notin E(G)[\mathcal{RF}(G), \mathcal{RF}(G)]$.*

Proof By (16) with $\beta = L(f)$, we have $\lambda_f(f) = L(f) > 0$, while Theorem 19 tells us that $\lambda_f(E(G)[\mathcal{RF}(G), \mathcal{RF}(G)]) = 0$. □

Corollary 22 *Let f be a test function. Then the centralizer $C_{\mathcal{RF}(G)}(f)$ of f in $\mathcal{RF}(G)$ satisfies*

$$C_{\mathcal{RF}(G)}(f) \cap E(G)[\mathcal{RF}(G), \mathcal{RF}(G)] = \{\mathbf{1}_G\}. \tag{25}$$

Proof If $f \in \mathcal{RF}(G)$ is cyclically reduced, of positive length, and normalized so that $f(0) = 1_G$, then, according to Theorem 10, the elements of $C_{\mathcal{RF}(G)}(f)$ are of the form

$$\big(f^k \circ f|_{[0,\alpha']}\big)^{\pm 1} \quad (k \in \mathbb{N}_0,\ 0 \le \alpha' \le L(f)),$$

with α' subject to certain further restrictions, which do not matter for the present purpose.

Now let f be a test function such that $f(0) = 1_G$. Then f is cyclically reduced by Lemma 12, and $L(f) > 0$ by definition, so that the above description of centralizer elements applies. Since the inverse of a test function is again a test function, Lemma 13 yields that, for f a normalized test function, $C_{\mathcal{RF}(G)}(f) \setminus \{\mathbf{1}_G\}$ consists entirely of test functions, and (25) follows in this case from Corollary 21. In general, we conjugate f by a G_0-element x to make $\tilde{f} = xfx^{-1}$ normalized, obtain (25) for $\tilde{f}$, and then conjugate back to obtain the same conclusion for f itself. □

5 Locally incompatible test functions

Definition 23 Two functions $f_1, f_2 \in \mathcal{F}(G)$ of lengths α_1 respectively α_2 are called *locally compatible* (loc. comp. for short), if there exist $\varepsilon > 0$ and points $\xi_i \in (0, \alpha_i)$ such that we either have

$$f_1(\xi_1 + \eta) = f_2(\xi_2 + \eta), \quad |\eta| < \varepsilon$$

or

$$f_1(\xi_1 + \eta) = f_2^{-1}(\xi_2 + \eta), \quad |\eta| < \varepsilon.$$

If f_1 and f_2 both have positive length, but are not locally compatible, they are called *locally incompatible* (loc. incomp. for short).

Local compatibility is clearly a symmetric relation on $\mathcal{F}(G)$; that is, we have

$$f_1 \text{ loc. comp. } f_2 \Longrightarrow f_2 \text{ loc. comp. } f_1 \quad (f_1, f_2 \in \mathcal{F}(G)). \tag{26}$$

A first observation concerning locally incompatible functions is as follows.

Lemma 24 (i) *If $f_1, f_2 \in \mathcal{F}(G)$ are locally incompatible, then $\varepsilon_0(f_1, f_2) = 0$.*

(ii) *If $f_1, f_2 \in \mathcal{F}(G)$ are locally incompatible, then so are the functions f_1^{-1} and f_2, as are the functions f_1^{-1} and f_2^{-1}.*

Proof (i) Let $L(f_i) = \alpha_i > 0$, and suppose that $\varepsilon_0(f_1, f_2) > 0$. Then there exists $\varepsilon > 0$ such that

$$f_1(\alpha_1 - \eta) f_2(\eta) = 1_G, \quad 0 \le \eta \le \varepsilon;$$

that is

$$f_2(\eta) = f_1^{-1}(\eta), \quad 0 \le \eta \le \varepsilon.$$

It follows that

$$f_2\Big(\frac{\varepsilon}{2} + \eta'\Big) = f_1^{-1}\Big(\frac{\varepsilon}{2} + \eta'\Big), \quad |\eta'| < \frac{\varepsilon}{2};$$

hence, f_2 is locally compatible to f_1, so f_1 is locally compatible to f_2 by (26), contradicting our hypothesis.

(ii) If f_1^{-1} and f_2 were locally compatible, we could find $\varepsilon > 0$ as well as points $\xi_1 \in (0, \alpha_1)$ and $\xi_2 \in (0, \alpha_2)$ with $L(f_i) = \alpha_i$ as above, such that

$$f_1^{-1}(\xi_1 + \eta) = f_2(\xi_2 + \eta), \quad |\eta| < \varepsilon$$

or

$$f_1^{-1}(\xi_1 + \eta) = f_2^{-1}(\xi_2 + \eta), \quad |\eta| < \varepsilon.$$

In the first case, we would conclude that

$$f_1(\alpha_1 - \xi_1 + \eta) = f_2^{-1}(\alpha_2 - \xi_2 + \eta), \quad |\eta| < \varepsilon,$$

while in the second case, we would find that

$$f_1(\alpha_1 - \xi_1 + \eta) = f_2(\alpha_2 - \xi_2 + \eta), \quad |\eta| < \varepsilon;$$

in both cases, it would thus follow that f_1 and f_2 are locally compatible, contradicting our hypothesis that f_1, f_2 are locally incompatible. The proof of the second assertion is similar. □

In the remainder of this section we are going to establish a somewhat technical result to the effect that every finite product of the form $\prod_j f_j^{\gamma_j}$ in pairwise locally incompatible test functions f_j is again a test function; cf. Proposition 29 at the end of this section. This proposition will be put to good use in the final section, where we derive certain structural properties of the groups $\mathcal{RF}(G)$ and their quotients $\mathcal{RF}(G)/E(G)$.

Lemma 25 *If f and g are locally incompatible test functions, then fg is again a test function.*

Proof Let $L(f) = \alpha$, $L(g) = \beta$, and set $h = fg$. By Lemma 24(i), we have $\gamma := L(h) = \alpha + \beta$, and

$$h(\xi) = \begin{Bmatrix} f(\xi), & 0 \le \xi < \alpha \\ f(\alpha)g(0), & \xi = \alpha \\ g(\xi - \alpha), & \alpha < \xi \le \gamma \end{Bmatrix} \quad (\xi \in [0, \gamma]). \tag{27}$$

Suppose for a contradiction that there exist $\varepsilon > 0$ and points $\xi_1, \xi_2 \in (0, \gamma)$, such that

$$h(\xi_1 + \eta) = h^{-1}(\xi_2 + \eta), \quad |\eta| < \varepsilon. \tag{28}$$

We may assume without loss of generality that $\xi_1 \neq \alpha$ and $\xi_2 \neq \beta$. Suppose first that $\xi_1 \in (0, \alpha)$. Then, for ε sufficiently small,

$$h(\xi_1 + \eta) = f(\xi_1 + \eta), \quad |\eta| < \varepsilon,$$

while

$$h^{-1}(\xi_2 + \eta) = \big(h(\gamma - \xi_2 - \eta)\big)^{-1} = \begin{Bmatrix} f^{-1}(\xi_2 - \beta + \eta), & \xi_2 > \beta \\ g^{-1}(\xi_2 + \eta), & \xi_2 < \beta \end{Bmatrix}, \quad |\eta| < \varepsilon.$$

We deduce from (28) that

$$f(\xi_1+\eta)=\left\{\begin{matrix} f^{-1}(\xi_2-\beta+\eta), & \xi_2>\beta \\ g^{-1}(\xi_2+\eta), & \xi_2<\beta \end{matrix}\right\}, \quad |\eta|<\varepsilon.$$

In the first case, the corresponding assertion contradicts the fact that f is a test function, while the assertion corresponding to the second case contradicts our hypothesis that f and g are locally incompatible. The case where $\xi_1 \in (\alpha,\gamma)$ is similar, and is omitted. □

Lemma 26 *Let f_1, f_2 be locally incompatible cyclically reduced functions, and let γ_1, γ_2 be non-zero integers. Then $f_1^{\gamma_1}$ and $f_2^{\gamma_2}$ are locally incompatible.*

Proof In view of the second part of Lemma 24, it is enough to consider the case when $\gamma_1, \gamma_2 \in \mathbb{N}$. Let $L(f_i)=\alpha_i$, and set $g := f_1^{\gamma_1}$ and $h := f_2^{\gamma_2}$. Since f_1, f_2 are of positive length and cyclically reduced, we have $L(g)=\gamma_1\alpha_1>0$ and $L(h)=\gamma_2\alpha_2>0$, and g is governed by the formula

$$g(\xi)=\left\{\begin{matrix} f_1(\xi \bmod \alpha_1), & \xi\notin\{\alpha_1,2\alpha_1,\ldots,\gamma_1\alpha_1\} \\ f_1(\alpha_1)f_1(0), & \xi\in\{\alpha_1,2\alpha_1,\ldots,(\gamma_1-1)\alpha_1\} \\ f_1(\alpha_1), & \xi=\gamma_1\alpha_1 \end{matrix}\right\} \quad (\xi\in[0,\gamma_1\alpha_1]),$$

with a corresponding formula holding for h; cf. (8). Suppose for a contradiction that there exist $\varepsilon>0$ and points $\xi_i \in (0,\gamma_i\alpha_i)$ such that either

$$g(\xi_1+\eta)=h(\xi_2+\eta), \quad |\eta|<\varepsilon \tag{29}$$

or

$$g(\xi_1+\eta)=h^{-1}(\xi_2+\eta), \quad |\eta|<\varepsilon. \tag{30}$$

We may clearly assume without loss of generality that $\xi_i \notin \{\alpha_i, 2\alpha_i, \ldots, (k_i-1)\alpha_i\}$ for $i=1,2$. Let

$$(j_1-1)\alpha_1<\xi_1<j_1\alpha_1$$

and

$$(j_2-1)\alpha_2<\xi_2<j_2\alpha_2$$

for integers j_1, j_2 satisfying $1\le j_1\le\gamma_1$ respectively $1\le j_2\le\gamma_2$. Then we have, for sufficiently small ε,

$$g(\xi_1+\eta)=f_1(\xi_1-(j_1-1)\alpha_1+\eta), \quad |\eta|<\varepsilon,$$

$$h(\xi_2+\eta)=f_2(\xi_2-(j_2-1)\alpha_2+\eta), \quad |\eta|<\varepsilon,$$

$$h^{-1}(\xi_2+\eta)=f_2^{-1}(\xi_2-(j_2-1)\alpha_2+\eta), \quad |\eta|<\varepsilon.$$

Hence, we find from (29) that

$$f_1(\xi_1-(j_1-1)\alpha_1+\eta)=f_2(\xi_2-(j_2-1)\alpha_2+\eta), \quad |\eta|<\varepsilon,$$

while (30) implies that

$$f_1(\xi_1 - (j_1 - 1)\alpha_1 + \eta) = f_2^{-1}(\xi_2 - (j_2 - 1)\alpha_2 + \eta), \quad |\eta| < \varepsilon,$$

both assertions contradicting our hypothesis that f_1 and f_2 are locally incompatible. □

Lemma 27 *Let f_1, f_2, f_3 be pairwise locally incompatible functions. Then $f_1 f_2$ and f_3 are locally incompatible.*

Proof Let $L(f_i) = \alpha_i > 0$, and set $g := f_1 f_2$. By the first part of Lemma 24,

$$\beta := L(g) = \alpha_1 + \alpha_2 > 0,$$

and

$$g(\xi) = \begin{Bmatrix} f_1(\xi), & 0 \leq \xi < \alpha_1 \\ f_1(\alpha_1) f_2(0), & \xi = \alpha_1 \\ f_2(\xi - \alpha_1), & \alpha_1 < \xi \leq \beta \end{Bmatrix} \quad (\xi \in [0, \beta]).$$

Suppose for a contradiction that there exist $\varepsilon > 0$ and points $\xi_1 \in (0, \beta)$, $\xi_2 \in (0, \alpha_3)$ such that either

$$g(\xi_1 + \eta) = f_3(\xi_2 + \eta), \quad |\eta| < \varepsilon \tag{31}$$

or

$$g(\xi_1 + \eta) = f_3^{-1}(\xi_2 + \eta), \quad |\eta| < \varepsilon. \tag{32}$$

We may assume without loss of generality that $\xi_1 \neq \alpha_1$, so that there are only two cases, according to whether $\xi_1 < \alpha_1$ or $\xi_1 > \alpha_1$. Suppose that $\xi_1 \in (0, \alpha_1)$. Then, for sufficiently small ε, we have

$$g(\xi_1 + \eta) = f_1(\xi_1 + \eta), \quad |\eta| < \varepsilon;$$

and (31), (32) imply that either

$$f_1(\xi_1 + \eta) = f_3(\xi_2 + \eta), \quad |\eta| < \varepsilon,$$

or that

$$f_1(\xi_1 + \eta) = f_3^{-1}(\xi_2 + \eta), \quad |\eta| < \varepsilon;$$

both assertions contradicting the fact that f_1 and f_3 are locally incompatible. The case where $\xi_1 \in (\alpha, \beta)$ is similar. □

As a final piece of preparation, we need to generalise Lemmas 25 and 27 to finitely many factors.

Lemma 28 (a) *Let $k \geq 2$ be an integer, and let $f_1, f_2, \ldots, f_k$ be pairwise locally incompatible functions. Then $f_1 f_2 \cdots f_{k-1}$ and f_k are locally incompatible.*

(b) *Let $f_1, f_2, \ldots, f_k$ be pairwise locally incompatible test functions, where $k \geq 1$. Then $f_1 f_2 \cdots f_k$ is again a test function.*

Proof (a) The assertion holds trivially for $k = 2$, and for $k = 3$ by Lemma 27. Let $k \geq 4$, suppose by way of induction that Assertion (a) holds with k replaced by $k - 1$, and let $f_1, f_2, \ldots, f_k \in \mathcal{F}(G)$ be pairwise locally incompatible. Then the functions $f_1 f_2 \cdots f_{k-2}$, f_{k-1}, and f_k are pairwise locally incompatible by the induction hypothesis applied to the sets $\{f_1, \ldots, f_{k-2}, f_{k-1}\}$ and $\{f_1, \ldots, f_{k-2}, f_k\}$, plus the fact that f_{k-1} and f_k are locally incompatible. By Lemma 27, $f_1 f_2 \cdots f_{k-1}$ and f_k are locally incompatible, completing the induction.

(b) This assertion holds trivially for $k = 1$, and for $k = 2$ by Lemma 25. Let $k \geq 3$, suppose by way of induction that Assertion (b) holds with k replaced by $k - 1$, and let $f_1, f_2, \ldots, f_k$ be pairwise locally incompatible test functions. By the induction hypothesis, $f_1 f_2 \cdots f_{k-1}$ is a test function, and, by Part (a), $f_1 f_2 \cdots f_{k-1}$ and f_k are locally incompatible; hence, by Lemma 25, $f_1 f_2 \cdots f_k$ is again a test function, completing the induction. □

We come to the main result of this section.

Proposition 29 *For $k \geq 1$, let $f_1, f_2, \ldots, f_k$ be pairwise locally incompatible test functions, and let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be non-zero integers. Then $f_1^{\gamma_1} f_2^{\gamma_2} \cdots f_k^{\gamma_k}$ is again a test function.*

Proof By Corollary 14 plus Lemmas 26 and 12, $f_1^{\gamma_1}, f_2^{\gamma_2}, \ldots, f_k^{\gamma_k}$ form a set of pairwise locally incompatible test functions. The result follows now from Part (b) of Lemma 28. □

6 Incompatible test functions with prescribed centralizer

6.1 An existence theorem

The purpose of this section is to establish an important and powerful existence theorem for families of pairwise locally incompatible test functions with prescribed centralizer, viz. Theorem 30 below. In what follows, we shall tacidly assume the axiom of choice, in particular as a hypothesis for the results of this section.

Theorem 30 *Let G be a non-trivial group, and let $0 < \Lambda < \mathbb{R}$ be any proper subgroup of the additive reals. Then there exists a family $\mathfrak{F}$ of pairwise locally incompatible normalized test functions in $\mathcal{RF}(G)$, such that $|\mathfrak{F}| = |G|^{(\mathbb{R}:\Lambda)}$, and such that the length function L induces an isomorphism $C_{\mathcal{RF}(G)}(f) \to \Lambda$ for each $f \in \mathfrak{F}$ in the sense of Theorem 10; that is, such that $\langle \Omega_f^0 \rangle = \Lambda$ for all $f \in \mathfrak{F}$.*

Before embarking on the proof of Theorem 30, we list a few consequences, which are important in their own right.

Corollary 31 *Suppose that G is non-trivial. Then there exists a family $\{f_\sigma\}_{\sigma \in S}$ of pairwise locally incompatible test functions in $\mathcal{RF}(G)$ with $|S| = |G|^{2^{\aleph_0}}$ and $L(f_\sigma) = \alpha_\sigma$ for each $\sigma \in S$, where $\{\alpha_\sigma\}_{\sigma \in S}$ is any given family of positive real numbers indexed by the elements of S.*

Proof Choose any subgroup Λ in Theorem 30 with $|\Lambda|$ countably infinite, to obtain a family $\{\bar{f}_\sigma\}_{\sigma \in S}$ of pairwise locally incompatible test functions, where $|S| = |G|^{2^{\aleph_0}}$. Scaling the functions $\bar{f}_\sigma$ appropriately, by setting

$$f_\sigma(\xi) := \bar{f}_\sigma(L(\bar{f}_\sigma)\xi\alpha_\sigma^{-1}) \quad (\sigma \in S,\ \xi \in [0, \alpha_\sigma]),$$

the family $\{f_\sigma\}_{\sigma\in S}$ meets the requirements of the corollary. □

Corollary 32 *Let G be a non-trivial group, and let A be a non-trivial torsion-free abelian group of rank at most $2^{\aleph_0}$. Then there exists a test function $f \in \mathcal{RF}(G)$ such that $C_{\mathcal{RF}(G)}(f) \cong A$.*

Proof Such a group A can be embedded into $\mathbb{R}$ as a proper subgroup Λ, $0 < \Lambda < \mathbb{R}$. Let $\mathfrak{F}$ be a family of test functions as described in Theorem 30 with respect to Λ. Then $\mathfrak{F} \neq \emptyset$, and every function $f \in \mathfrak{F}$ satisfies $C_{\mathcal{RF}(G)}(f) \cong \Lambda \cong A$. □

Corollary 33 *We have $|\mathcal{RF}(G)| = |G|^{2^{\aleph_0}}$.*

Proof Since $\mathcal{RF}(G)$ is a subset of $\mathcal{F}(G)$, and, by definition,

$$|\mathcal{F}(G)| = |G_0| + \sum_{\alpha\in\mathbb{R}_{>0}} |G|^{2^{\aleph_0}} = |G| + 2^{\aleph_0}\cdot|G|^{2^{\aleph_0}} = |G|^{2^{\aleph_0}},$$

we have

$$|\mathcal{RF}(G)| \leq |G|^{2^{\aleph_0}}; \tag{33}$$

cf., for instance [8], in particular Chap. X, § 4. For the reverse inequality,

$$|\mathcal{RF}(G)| \geq |G|^{2^{\aleph_0}}, \tag{34}$$

we may assume that G is non-trivial, since (34) holds trivially for $G = \{1_G\}$. However, if G is non-trivial, Inequality (34) follows immediately from Corollary 31 together with the fact, implied by Lemma 12, that test functions are reduced. Inequalities (33) and (34) together now yield our claim, since we assume the axiom of choice. □

The proof of Theorem 30 will occupy the remainder of this section.

6.2 An arithmetic lemma

The following purely arithmetic result will be needed in Sect. 6.3.

Lemma 34 *Let ξ_1, ξ_2 be real numbers, one of which is rational. Suppose that there exists $\varepsilon > 0$ such that*

$$(\xi_1+\eta)^2 \in \mathbb{Q} \iff (\xi_2+\eta)^2 \in \mathbb{Q}, \quad \eta \in (0,\varepsilon)\cap\bar{\mathbb{Q}}, \tag{35}$$

where $\bar{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$. Then $\xi_1 = \xi_2$.

Proof It is enough to establish the desired conclusion (that $\xi_1 = \xi_2$) under the (formally stronger) hypothesis that one of ξ_1, ξ_2 is rational, and that

$$(\xi_1+\eta)^2 \in \mathbb{Q} \iff (\xi_2+\eta)^2 \in \mathbb{Q}, \quad \eta \in (-\varepsilon,\varepsilon)\cap\bar{\mathbb{Q}}. \tag{36}$$

Indeed, suppose that ξ_1, ξ_2 meet the hypotheses of Lemma 34. Then, choosing $\eta_1 \in (0, \frac{\varepsilon}{2}] \cap \mathbb{Q}$, and setting $\xi_i' := \xi_i + \eta_1$, one of ξ_1', ξ_2' is still rational, and we have

$$(\xi_1' + \eta)^2 \in \mathbb{Q} \iff (\xi_2' + \eta)^2 \in \mathbb{Q}, \quad \eta \in (-\eta_1, \eta_1) \cap \bar{\mathbb{Q}}.$$

The (formally) weaker version of Lemma 34 now yields $\xi_1' = \xi_2'$, hence also $\xi_1 = \xi_2$.

We now prove this formally weaker version of Lemma 34. Suppose without loss of generality that ξ_1 is rational. If q is a rational number satisfying $0 < q < \varepsilon$, then $(\xi_1 \pm q)^2 \in \mathbb{Q}$. Invoking Condition (36) with $\eta = q$ and $\eta = -q$, we find that $(\xi_2 + q)^2, (\xi_2 - q)^2 \in \mathbb{Q}$. Subtracting yields that $4q\xi_2$ is rational, thus $\xi_2 \in \mathbb{Q}$, since q is rational and non-zero.

Now let r be a rational number such that $r \neq 0$ and $r\sqrt{2} \in (\xi_1, \xi_1 + \varepsilon)$. Applying Condition (36) with

$$\eta = r\sqrt{2} - \xi_1 \in \bar{\mathbb{Q}} \cap (0, \varepsilon),$$

we obtain that

$$(\xi_2 + r\sqrt{2} - \xi_1)^2 = (\xi_2 - \xi_1)^2 + 2r^2 + 2r(\xi_2 - \xi_1)\sqrt{2} \in \mathbb{Q}.$$

Since $r \neq 0$, the assumption that $\xi_1 \neq \xi_2$ implies that $\sqrt{2}$ is rational, a contradiction. Hence, $\xi_1 = \xi_2$, as claimed. □

Corollary 35 *Let ξ_1, ξ_2 be real numbers, and suppose there exists $\varepsilon > 0$ such that*

$$(\xi_1 + \eta)^2 \in \mathbb{Q} \iff (\xi_2 + \eta)^2 \in \mathbb{Q}, \quad \eta \in (0, \varepsilon).$$

Then we have $\xi_1 = \xi_2$.

Proof Choose $\eta_1 \in (0, \varepsilon)$ such that $\xi_1' := \xi_1 + \eta_1$ is rational, and set $\xi_2' := \xi_2 + \eta_1$. Then ξ_1' and ξ_2' satisfy

$$(\xi_1' + \eta)^2 \in \mathbb{Q} \iff (\xi_2' + \eta)^2 \in \mathbb{Q}, \quad \eta \in (0, \varepsilon - \eta_1).$$

By Lemma 34, we have $\xi_1' = \xi_2'$, hence also $\xi_1 = \xi_2$. □

The next two subsections take up the proof of Theorem 30, distinguishing cases as to whether Λ is cyclic or dense in $\mathbb{R}$.[4]

6.3 The case where Λ is cyclic

Let α be any positive real number, and set $\Lambda := \langle \alpha \rangle$. We are going to construct a family $\mathfrak{F}$ of pairwise locally incompatible normalized test functions in $\mathcal{RF}(G)$ of size $|\mathfrak{F}| = |G|^{2^{\aleph_0}}$, such that $\langle \Omega_f^0 \rangle = \Lambda$ for each $f \in \mathfrak{F}$, where Λ is as above.

Let x be a non-trivial element of the group G, and define a function

$$g : ([0, \alpha) \cap \bar{\mathbb{Q}}) \cup \{\alpha\} \to G$$

[4] The reader should recall that every subgroup of the additive reals is either cyclic or dense; cf., for instance, [2, Chap. 1, Lemma 1.3].

via

$$g(\xi) := \begin{Bmatrix} x, & \xi^2 \in \mathbb{Q} \text{ and } \xi \notin \{0, \alpha\} \\ 1_G, & \text{otherwise} \end{Bmatrix} \quad (\xi \in ([0,\alpha) \cap \bar{\mathbb{Q}}) \cup \{\alpha\}),$$

where, as before, $\bar{\mathbb{Q}}$ denotes the set of real numbers which are algebraic over $\mathbb{Q}$. Moreover, introduce an equivalence relation on $(0,\alpha)$ by setting

$$\xi_1 \sim \xi_2 :\Longleftrightarrow \xi_1 - \xi_2 \in \bar{\mathbb{Q}},$$

and let $\mathfrak{T}$ be a complete set of representatives for the quotient

$$\big((0,\alpha) - \bar{\mathbb{Q}}\big)/\sim.$$

Clearly, $|\mathfrak{T}| = 2^{\aleph_0}$. Let $h : \mathfrak{T} \to G$ be an arbitrary (set-theoretic) map, and define a function $f_h : [0,\alpha] \to G$ via

$$f_h(\xi) := \begin{Bmatrix} g(\xi), & \xi \in ([0,\alpha) \cap \bar{\mathbb{Q}}) \cup \{\alpha\} \\ h(\tau), & \xi \sim \tau \; (\tau \in \mathfrak{T}) \end{Bmatrix} \quad (0 \le \xi \le \alpha);$$

in particular,

$$f_h(0) = g(0) = 1_G.$$

We claim that every function f_h obtained in this way is in fact a test function, and that f_{h_1} and f_{h_2} are locally incompatible for any two maps $h_1, h_2 : \mathfrak{T} \to G$ such that $h_1 \neq h_2$.

Suppose that, for some map $h : \mathfrak{T} \to G$, the function f_h is not a test function. Then there exist $\varepsilon > 0$ and points $\xi_1, \xi_2 \in (0,\alpha)$ such that

$$f_h(\xi_1 + \eta) = f_h^{-1}(\xi_2 + \eta), \quad |\eta| < \varepsilon. \tag{37}$$

We may assume without loss of generality that ξ_1 is rational. As the set $\{1, 2^{1/3}, 2^{2/3}\}$ is linearly independent over $\mathbb{Q}$, there exists a positive integer n, such that $\sqrt[3]{2}/n < \varepsilon$ and $(\sqrt[3]{2}/n + \xi_1)^2 \notin \mathbb{Q}$. Since we have $\sqrt[3]{2}/n + \xi_1 \in \bar{\mathbb{Q}} \cap [0,1]$, but $(\sqrt[3]{2}/n + \xi_1)^2 \notin \mathbb{Q}$, the definitions of the functions f_h and g together imply that

$$f_h(\sqrt[3]{2}/n + \xi_1) = g(\sqrt[3]{2}/n + \xi_1) = 1_G \neq x = g(\xi_1) = f_h(\xi_1).$$

Using (37), it follows that

$$f_h(\alpha - \xi_2) \neq f_h(\alpha - \xi_2 - \sqrt[3]{2}/n);$$

however, since on $(0,\alpha) - \bar{\mathbb{Q}}$ the function f_h is constant on cosets modulo $\bar{\mathbb{Q}}$, this implies that $\alpha - \xi_2 \in \bar{\mathbb{Q}}$. If $x^2 \neq 1_G$, setting $\eta = 0$ in (37) immediately gives a contradiction; while for $x^2 = 1_G$, we find that

$$(\xi_1 + \eta)^2 \in \mathbb{Q} \Longleftrightarrow (\xi_2 - \alpha + \eta)^2 \in \mathbb{Q}, \quad \eta \in (-\varepsilon, \varepsilon) \cap \bar{\mathbb{Q}}.$$

It follows from Lemma 34 that $\xi_1 = \xi_2 - \alpha$, which is impossible, since $\xi_1, \xi_2 \in (0,\alpha)$. Hence, f_h is a test function, as claimed.

Next, suppose that f_{h_1} and f_{h_2} are locally compatible, where $h_1, h_2 \in G^{\mathfrak{T}}$. Then there exist $\varepsilon > 0$ and points $\xi_1, \xi_2 \in (0, \alpha)$ such that either

$$f_{h_1}(\xi_1 + \eta) = f_{h_2}(\xi_2 + \eta), \quad |\eta| < \varepsilon \tag{38}$$

or

$$f_{h_1}(\xi_1 + \eta) = f_{h_2}^{-1}(\xi_2 + \eta), \quad |\eta| < \varepsilon. \tag{39}$$

Let us first consider (38). We may again assume that ξ_1 is rational; and, arguing in a similar way as before, we obtain that ξ_2 is algebraic. This yields

$$(\xi_1 + \eta)^2 \in \mathbb{Q} \iff (\xi_2 + \eta)^2 \in \mathbb{Q}, \quad \eta \in (-\varepsilon, \varepsilon) \cap \bar{\mathbb{Q}},$$

implying $\xi_1 = \xi_2$ by Lemma 34. Let $\tau \in \mathfrak{T}$ be an arbitrary representative. Since $\bar{\mathbb{Q}}$ is dense in $\mathbb{R}$, there exists a real number t such that $t \sim \tau$ and $|t - \xi_1| < \varepsilon$. The equation

$$f_{h_1}(t) = f_{h_1}(\xi_1 + (t - \xi_1)) = f_{h_2}(\xi_1 + (t - \xi_1)) = f_{h_2}(t)$$

derived from (38) now shows that $h_1(\tau) = h_2(\tau)$, thus $h_1 = h_2$, since $\tau \in \mathfrak{T}$ was arbitrary.

Now suppose that (39) applies. Again, we may assume that ξ_1 is rational, and we obtain as before that $\alpha - \xi_2 \in \bar{\mathbb{Q}}$. This yields

$$g(\xi_1 + \eta) = \big(g(\alpha - \xi_2 - \eta)\big)^{-1}, \quad \eta \in (-\varepsilon, \varepsilon) \cap \bar{\mathbb{Q}}. \tag{40}$$

Choosing η in (40) small and rational shows that we must have $x^2 = 1_G$, and we now find from (40) that

$$(\xi_1 + \eta)^2 \in \mathbb{Q} \iff (\xi_2 - \alpha + \eta)^2 \in \mathbb{Q}, \quad \eta \in (-\varepsilon, \varepsilon) \cap \bar{\mathbb{Q}}.$$

Applying Lemma 34, we get that $\xi_1 = \xi_2 - \alpha$, which again is impossible, since the left-hand side is positive, while the right-hand side is negative. Hence, (39) does not arise, while (38) forces $h_1 = h_2$.

All in all, we have shown that the family of functions $\mathfrak{F} = \{f_h\}_{h \in G^{\mathfrak{T}}}$ consists of pairwise locally incompatible normalized test functions of length α; and since $|\mathfrak{T}| = 2^{\aleph_0}$, we have $|\mathfrak{F}| = |G|^{2^{\aleph_0}}$, as required.

It only remains to check the assertion concerning centralizers. Since each $f_h \in \mathfrak{F}$ is cyclically reduced, of positive length, and normalized, the length function L on $\mathcal{RF}(G)$, according to Theorem 10, induces isomorphisms $C_{\mathcal{RF}(G)}(f_h) \to \langle \Omega^0_{f_h} \rangle$ for all functions $h : \mathfrak{T} \to G$; thus, we have to determine the set $\Omega^0_{f_h}$ of strong periods for $h \in G^{\mathfrak{T}}$.

By definition,

$$\{0, \alpha\} \subseteq \Omega^0_{f_h}, \quad h \in G^{\mathfrak{T}}.$$

Suppose that there exists a period $\omega \in \Omega_{f_h}$ with $0 < \omega < \alpha$. If ω is algebraic, then we have

$$g(\xi) = f_h(\xi) = f_h(\xi + \omega) = g(\xi + \omega), \quad \xi \in (0, \alpha - \omega) \cap \bar{\mathbb{Q}},$$

which is equivalent to

$$\xi^2 \in \mathbb{Q} \iff (\omega + \xi)^2 \in \mathbb{Q}, \quad \xi \in (0, \alpha - \omega) \cap \bar{\mathbb{Q}}.$$

From Lemma 34, we infer that $\omega = 0$, a contradiction.

Now suppose that ω is transcendental, and let $\tau_\omega \in \mathfrak{T}$ be such that $\omega \sim \tau_\omega$. Then we have

$$g(\xi) = f_h(\xi) = f_h(\xi + \omega) = h(\tau_\omega), \quad \xi \in (0, \alpha - \omega) \cap \bar{\mathbb{Q}};$$

that is, g would have to be constant on the algebraic points of the interval $(0, \alpha - \omega)$, which is clearly impossible: for instance, pick ξ_1 rational and $\xi_2 = \sqrt[3]{2}/n$ for $n \in \mathbb{N}$ sufficiently large. It follows that

$$\Omega^0_{f_h} \subseteq \Omega_{f_h} \subseteq \{0, \alpha\} \subseteq \Omega^0_{f_h},$$

and so

$$\left\langle \Omega^0_{f_h} \right\rangle = \langle \alpha \rangle = \Lambda, \quad h \in G^{\mathfrak{T}},$$

as required.

6.4 The case when Λ is dense

Since $(\mathbb{R}, +)$ has no maximal subgroups, we can find a subgroup Λ' such that $\Lambda < \Lambda' < \mathbb{R}$ and $(\Lambda' : \Lambda) \leq \aleph_0$. Let x be any fixed non-trivial element of G, α a fixed positive element of Λ, and let ξ_0 be a fixed real number in $(\mathbb{R} - \Lambda') \cap [0, \alpha]$.

Consider functions $f : [0, \alpha] \to G$ satisfying the following three conditions.

(i) f is constant on each non-trivial coset of Λ' in $\mathbb{R}$.
(ii) We have

$$f(\xi) = \begin{Bmatrix} 1_G, & \xi \in \Lambda \\ x, & \xi \in \Lambda' - \Lambda \end{Bmatrix} \quad (\xi \in \Lambda' \cap [0, \alpha]);$$

in particular, $f(0) = 1_G$.
(iii) We have $f(\xi_0/2) = 1_G$ and $f(\alpha - \xi_0/2) = x$.

Since $\xi_0 \notin \Lambda'$, we have $\xi_0/2, -\xi_0/2 \notin \Lambda'$; moreover, these points are in distinct cosets modulo Λ'. Hence, the third condition does not conflict with Conditions (i) and (ii). Denote by $\mathfrak{F}$ the family of all functions $f : [0, \alpha] \to G$ meeting Conditions (i)–(iii).

The number $|\mathfrak{F}|$ of such functions f equals the number of functions

$$\mathbb{R}/\Lambda' - \big\{[0], [\xi_0/2], [-\xi_0/2]\big\} \longrightarrow G;$$

and, since $(\mathbb{R} : \Lambda') = \infty$, the three missing cosets do not change the cardinality of the domain, thus there are $|G|^{(\mathbb{R}:\Lambda')}$ such functions. Moreover, as Λ' was chosen such that $(\Lambda' : \Lambda) \leq \aleph_0$, we have $(\mathbb{R} : \Lambda') = (\mathbb{R} : \Lambda)$, so that

$$|\mathfrak{F}| = |G|^{(\mathbb{R}:\Lambda')} = |G|^{(\mathbb{R}:\Lambda)},$$

as required.

We claim that each function $f \in \mathfrak{F}$ is a test function, and that any two different such functions are locally incompatible.

Suppose that $f \in \mathfrak{F}$ is not a test function. Then there exist $\varepsilon > 0$ and points $\xi_1, \xi_2 \in (0, \alpha)$, such that

$$f(\xi_1 + \eta) = f^{-1}(\xi_2 + \eta), \quad |\eta| < \varepsilon. \tag{41}$$

Since Λ is dense in $\mathbb{R}$, we may assume without loss of generality that $\xi_1 \in \Lambda$. Further, since $\Lambda' - \Lambda$ is dense in $\mathbb{R}$, we can choose $\eta_1 \in (-\varepsilon, \varepsilon)$ such that $\xi_1 + \eta_1 \in \Lambda' - \Lambda$; thus, applying (41) together with Condition (ii), we get that

$$f(\xi_2' - \eta_1) = f(\alpha - \xi_2 - \eta_1) = \big(f^{-1}(\xi_2 + \eta_1)\big)^{-1} = \big(f(\xi_1 + \eta_1)\big)^{-1} = x^{-1},$$

where $\xi_2' := \alpha - \xi_2$. Making again use of (41) and (ii), it follows that

$$f(\xi_2') = \big(f^{-1}(\xi_2)\big)^{-1} = \big(f(\xi_1)\big)^{-1} = 1_G \neq x^{-1} = f(\xi_2' - \eta_1),$$

implying that f is not constant on the coset $\xi_2' + \Lambda'$. Since, by Conditions (i) and (ii), the only coset modulo Λ', on which f is not constant, is Λ' itself, we deduce that $\xi_2' \in \Lambda'$, therefore also $\xi_2 \in \Lambda'$, since $\alpha \in \Lambda$. Using the fact that Λ' is dense in $\mathbb{R}$, we can now choose $\eta_2 \in (-\varepsilon, \varepsilon)$ such that $\xi_1 + \eta_2 \in \xi_0/2 + \Lambda'$. Then, by (41) together with Conditions (i) and (iii), we have

$$1_G = f(\xi_0/2) = f(\xi_1 + \eta_2) = f^{-1}(\xi_2 + \eta_2) = \big(f(\xi_2' - \eta_2)\big)^{-1} = \big(f(\alpha - \xi_0/2)\big)^{-1} = x^{-1},$$

a contradiction. Hence, f is a test function, as claimed.

Now suppose that $f_1, f_2 \in \mathfrak{F}$ are locally compatible. Then there exist $\varepsilon > 0$ and points $\xi_1, \xi_2 \in (0, \alpha)$, such that either

$$f_1(\xi_1 + \eta) = f_2(\xi_2 + \eta), \quad |\eta| < \varepsilon,$$

or

$$f_1(\xi_1 + \eta) = f_2^{-1}(\xi_2 + \eta), \quad |\eta| < \varepsilon.$$

As before, we may assume that $\xi_1 \in \Lambda$; and reasoning similar to the one above then shows that $\xi_2 \in \Lambda'$. Setting $\eta = 0$, it follows further that $\xi_2 \in \Lambda$ and, since f_1, f_2 are constant on cosets modulo Λ, we may suppose that $\xi_1 = \xi_2$. First consider the case when

$$f_1(\xi_1 + \eta) = f_2(\xi_1 + \eta), \quad |\eta| < \varepsilon.$$

Then f_1 and f_2 coincide on an open interval, and are periodic with a dense set of periods; hence, they are equal. Next suppose that

$$f_1(\xi_1 + \eta) = f_2^{-1}(\xi_1 + \eta), \quad |\eta| < \varepsilon.$$

Again choosing $\eta_2 \in (-\varepsilon, \varepsilon)$ such that $\xi_1 + \eta_2 \in \xi_0/2 + \Lambda'$, we find that

$$\begin{aligned} 1_G &= f_1(\xi_0/2) = f_1(\xi_1 + \eta_2) = f_2^{-1}(\xi_1 + \eta_2) = \big(f_2(\alpha - \xi_1 - \eta_2)\big)^{-1} \\ &= \big(f_2(\alpha - \xi_0/2)\big)^{-1} = x^{-1}, \end{aligned}$$

a contradiction. Hence, the second case does not arise, while the first case implies $f_1 = f_2$.

So far, we have shown that $\mathfrak{F}$ is a family of pairwise locally incompatible normalized test functions of length $\alpha \in \Lambda$, and that $|\mathfrak{F}| = |G|^{(\mathbb{R}:\Lambda)}$. It only remains to verify that the group generated by the strong periods of $f \in \mathfrak{F}$ coincides with Λ. Since f is constant on cosets modulo Λ, every element in the set $\Lambda \cap [0, \alpha]$ is a period of f. Moreover, since $\alpha \in \Lambda$, every $\omega \in \Lambda \cap [0, \alpha]$ is also a strong period. As Λ is dense in $\mathbb{R}$, it is generated by every intersection with an open interval; hence, the strong periods generate a subgroup of $\mathbb{R}$

containing Λ. Now let $\omega < \alpha$ be a period of f, and choose points $\xi_1, \xi_2 \in (0, \alpha - \omega)$ with $\xi_1 \in \Lambda$ and $\xi_2 \in \Lambda' - \Lambda$. Then

$$f(\xi_1 + \omega) = f(\xi_1) = 1_G,$$

while

$$f(\xi_2 + \omega) = f(\xi_2) = x.$$

Since $\xi_1 - \xi_2 \in \Lambda'$, the points $\xi_1 + \omega$ and $\xi_2 + \omega$ lie in the same coset modulo Λ'; and, since f is constant on Λ'-cosets different from Λ' itself, we conclude that $\omega \in \Lambda'$. Moreover, since $f(\xi_1 + \omega) = 1_G$, we have $\omega \in \Lambda$. We conclude that every period of f is in fact contained in Λ; hence,

$$\Lambda \leq \langle \Omega_f^0 \rangle \leq \langle \Omega_f \rangle \leq \Lambda,$$

and so

$$\langle \Omega_f^0 \rangle = \Lambda,$$

as desired.

7 The cardinality of $\mathcal{RF}(G)$ revisited

Here we give a second proof of Corollary 33 concerning the cardinality of the group $\mathcal{RF}(G)$, this time based on properties of the Cantor discontinuum.

We may assume that G is non-trivial, and may concentrate on the proof of (34). In order to establish the latter inequality, we shall produce a set of reduced functions of cardinality $|G|^{2^{\aleph_0}}$ as follows. Let $\mathcal{C} \subset [0, 1]$ be the Cantor discontinuum; that is, the set of real numbers in $[0, 1]$ whose triadic expansion avoids the digit 1. It is well known that $\mathcal{C}$ is compact, of cardinality $|\mathcal{C}| = 2^{\aleph_0}$, and that $[0, 1] \setminus \mathcal{C}$ is dense in $[0, 1]$. We shall construct a map $f : [0, 1] \setminus \mathcal{C} \to G$, such that, for every (set-theoretic) function $g : \mathcal{C} \to G$, the map $f \oplus g : [0, 1] \to G$ given by

$$(f \oplus g)(\xi) := \begin{Bmatrix} f(\xi), & \xi \notin \mathcal{C} \\ g(\xi), & \xi \in \mathcal{C} \end{Bmatrix} \quad (0 \leq \xi \leq 1)$$

is reduced. In this way, Inequality (34) follows and, together with (33), again establishes Corollary 33.

Let $\{C_\mu\}_{\mu \in \mathbb{N}_0}$ be an enumeration of the connected components of $[0, 1] \setminus \mathcal{C}$, $C_\mu = (a_\mu, b_\mu)$ say, let $\{\alpha_\mu\}_{\mu \in \mathbb{N}_0}$ be the corresponding sequence of interval lengths, $\alpha_\mu = b_\mu - a_\mu$, and let $\{f_\mu\}_{\mu \in \mathbb{N}_0}$ be a sequence of pairwise locally incompatible test functions such that $L(f_\mu) = \alpha_\mu$ for all $\mu \in \mathbb{N}_0$ (such a sequence exists by Corollary 31, since G is non-trivial). Then we define a function f on $[0, 1] \setminus \mathcal{C}$ via

$$f(\xi) := f_\mu(\xi - a_\mu), \quad \xi \in C_\mu.$$

Now let $g : \mathcal{C} \to G$ be an arbitrary map. We claim that $f \oplus g : [0, 1] \to G$ is reduced. Suppose otherwise. Then there exists $\xi_0 \in (0, 1)$ and $\varepsilon > 0$ such that $(f \oplus g)(\xi_0) = 1_G$, and

$$(f \oplus g)(\xi_0 + \eta)(f \oplus g)(\xi_0 - \eta) = 1_G, \quad 0 < \eta < \varepsilon. \tag{42}$$

Suppose first that $\xi_0 \notin \mathcal{C}$. Then $\xi_0 \in C_\mu$ for some $\mu \in \mathbb{N}_0$, and we find that

$$\left\{ \begin{array}{l} f_\mu(\xi_0 - a_\mu) = 1_G \\ f_\mu(\xi_0 - a_\mu + \eta) f_\mu(\xi_0 - a_\mu - \eta) = 1_G, \quad 0 < \eta < \varepsilon' \end{array} \right\}, \tag{43}$$

where $\varepsilon' := \min\{\varepsilon, \xi_0 - a_\mu, b_\mu - \xi_0\}$. Assertions (43) imply however that f_μ is not reduced, contradicting the fact that f_μ is a test function in view of Lemma 12.

Now suppose that $\xi_0 \in \mathcal{C}$. Then we claim that there exist points $\xi_1, \xi_2 \in (0, 1)$ satisfying

(i) $0 < \xi_0 - \varepsilon < \xi_1 < \xi_0 < \xi_2 < \xi_0 + \varepsilon < 1$,

(ii) $|\xi_1 - \xi_0| = |\xi_2 - \xi_0|$,

(iii) $\xi_1, \xi_2 \notin \mathcal{C}$.

Indeed, by density of $[0, 1] \setminus \mathcal{C}$ in $[0, 1]$, there exists ξ_1' such that $\xi_0 - \varepsilon < \xi_1' < \xi_0$ and $\xi_1' \notin \mathcal{C}$. Let $\xi_1' \in C_{\mu_1}$, choose $\varepsilon'' > 0$ such that

$$(\xi_1' - \varepsilon'', \xi_1' + \varepsilon'') \subseteq C_{\mu_1} \cap (\xi_0 - \varepsilon, \xi_0),$$

and let $\xi_2' := \xi_0 + |\xi_1' - \xi_0|$. Again making use of the density property of $[0, 1] \setminus \mathcal{C}$, there exists a point $\xi_2 \notin \mathcal{C}$ such that $\xi_2' - \varepsilon'' < \xi_2 < \xi_2'$. Let $\xi_1 := \xi_1' + |\xi_2 - \xi_2'|$. Then $\xi_1 \in C_{\mu_1}$, in particular, $\xi_1 \notin \mathcal{C}$, and the points ξ_1, ξ_2 satisfy Properties (i)–(iii) by construction.

We have $\xi_i \in C_{\mu_i}$ $(i = 1, 2)$ for certain indices $\mu_1, \mu_2 \in \mathbb{N}_0$ such that $\mu_1 \neq \mu_2$, and (42) now implies that

$$f_{\mu_1}(\xi_1 - a_{\mu_1} + \eta) = f_{\mu_2}^{-1}(\alpha_{\mu_2} - \xi_2 + a_{\mu_2} + \eta), \quad |\eta| < \tilde{\varepsilon}, \tag{44}$$

where $\tilde{\varepsilon} > 0$ is chosen such that

$$\left\{ \begin{array}{l} (\xi_1 - \tilde{\varepsilon}, \xi_1 + \tilde{\varepsilon}) \subseteq (\xi_0 - \varepsilon, \xi_0) \cap C_{\mu_1} \\ (\xi_2 - \tilde{\varepsilon}, \xi_2 + \tilde{\varepsilon}) \subseteq (\xi_0, \xi_0 + \varepsilon) \cap C_{\mu_2} \end{array} \right\}.$$

However, (44) implies that f_{μ_1} loc. comp. f_{μ_2}, contradicting the choice of the sequence $\{f_\mu\}$, since $\mu_1 \neq \mu_2$. Hence, $f \oplus g$ is reduced as claimed.

8 A structure theorem

The aim of this final section is to establish a number of structural properties of $\mathcal{RF}(G)$ and its quotient group $\mathcal{RF}_0(G) := \mathcal{RF}(G)/E(G)$. In what follows, $^-$ denotes abelianization, and the homomorphisms

$$\begin{aligned} \mathfrak{ab}(G) &: \mathcal{RF}(G) \to \overline{\mathcal{RF}(G)} \\ \mathfrak{ab}_0(G) &: \mathcal{RF}(G) \to \mathcal{RF}(G) \big/ (E(G)[\mathcal{RF}(G), \mathcal{RF}(G)]) \\ \pi &: \mathcal{RF}(G) \to \mathcal{RF}_0(G) \end{aligned}$$

are the canonical ones. As a useful piece of general nonsense, we note that, for a group Γ, a normal subgroup $\Delta \trianglelefteq \Gamma$, and a verbal functor $V_{\mathcal{W}}(\cdot) : \mathbf{Groups} \to \mathbf{Groups}$ on the category **Groups** of groups and homomorphisms associated with a set

$$\mathcal{W} = \{w_\mu(x_\nu) : \mu \in M\}$$

of words in the variables x_ν, we have a canonical isomorphism

$$\Gamma\big/(\Delta V_{\mathcal{W}}(\Gamma)) \cong (\Gamma/\Delta)\big/V_{\mathcal{W}}(\Gamma/\Delta).$$

This follows immediately from the canonical isomorphism

$$\Gamma\big/(\Delta V_{\mathcal{W}}(\Gamma)) \cong \Gamma/\Delta\big/\Delta V_{\mathcal{W}}(\Gamma)/\Delta$$

plus the trivial fact that

$$\Delta V_{\mathcal{W}}(\Gamma)/\Delta = V_{\mathcal{W}}(\Gamma/\Delta).$$

As a special case, we have a canonical isomorphism

$$\mathcal{RF}(G)\big/(E(G)[\mathcal{RF}(G), \mathcal{RF}(G)]) \cong \overline{\mathcal{RF}_0(G)},$$

which we shall tacitly use to view $\mathfrak{ab}_0(G)$ as a map

$$\mathfrak{ab}_0(G) : \mathcal{RF}(G) \to \overline{\mathcal{RF}_0(G)}.$$

The following crucial result analyses the subgroup generated by a set of incompatible test functions, as well as the images of this subgroup under the projections π, $\mathfrak{ab}(G)$, and $\mathfrak{ab}_0(G)$.

Proposition 36 *Let $\{f_\sigma\}_{\sigma\in S}$ be a set of pairwise locally incompatible test functions. Then we have the following.*

(i) *The subgroup $\mathfrak{F}_S := \langle f_\sigma : \sigma \in S\rangle$ of $\mathcal{RF}(G)$ is free with basis $\{f_\sigma\}_{\sigma\in S}$, and satisfies*

$$\mathfrak{F}_S \cap E(G) = \big\{\mathbf{1}_G\big\}; \tag{45}$$

in particular, $\mathfrak{F}_S$ is hyperbolic, and the image of $\mathfrak{F}_S$ under the projection π is free with basis $\{\pi(f_\sigma)\}_{\sigma\in S}$.

(ii) *The image of $\mathfrak{F}_S$ under the projections $\mathfrak{ab}(G)$ respectively $\mathfrak{ab}_0(G)$ is free abelian with basis $\{\mathfrak{ab}(G)(f_\sigma)\}_{\sigma\in S}$ and $\{\mathfrak{ab}_0(G)(f_\sigma)\}_{\sigma\in S}$, respectively.*

Proof We first establish Part (ii). Let $\mathfrak{N}$ denote either of the normal subgroups $[\mathcal{RF}(G), \mathcal{RF}(G)]$ or $E(G)[\mathcal{RF}(G), \mathcal{RF}(G)]$, and consider a relation

$$(f_{\sigma_1}\mathfrak{N})^{\gamma_1}(f_{\sigma_2}\mathfrak{N})^{\gamma_2}\cdots(f_{\sigma_r}\mathfrak{N})^{\gamma_r} = (f_{\sigma_1}^{\gamma_1}f_{\sigma_2}^{\gamma_2}\cdots f_{\sigma_r}^{\gamma_r})\mathfrak{N} = \mathfrak{N} \tag{46}$$

in $\mathfrak{ab}(G)(\mathfrak{F}_S)$ respectively $\mathfrak{ab}_0(G)(\mathfrak{F}_S)$ with $r \geq 0$, distinct indices $\sigma_1, \sigma_2, \ldots, \sigma_r$, and exponents $\gamma_1, \gamma_2, \ldots, \gamma_r \in \mathbb{Z}\setminus\{0\}$. If $r > 0$, then the function $f_{\sigma_1}^{\gamma_1}f_{\sigma_2}^{\gamma_2}\cdots f_{\sigma_r}^{\gamma_r}$ is a test function by Proposition 29, thus $f_{\sigma_1}^{\gamma_1}f_{\sigma_2}^{\gamma_2}\cdots f_{\sigma_r}^{\gamma_r} \notin \mathfrak{N}$ by Corollary 21, contradicting (46). Hence, we must have $r = 0$, so $\mathfrak{ab}(G)(\mathfrak{F}_S)$ and $\mathfrak{ab}_0(G)(\mathfrak{F}_S)$ are indeed free abelian groups of rank $|S|$, freely generated by the sets $\{\mathfrak{ab}(G)(f_\sigma)\}_{\sigma\in S}$ respectively $\{\mathfrak{ab}_0(G)(f_\sigma)\}_{\sigma\in S}$.

Next, we show that the group $\mathfrak{F}_S$ itself is free with basis $\{f_\sigma\}_{\sigma\in S}$. Consider a non-empty reduced word $w(f_\sigma) = f_{\sigma_1}^{\gamma_1}f_{\sigma_2}^{\gamma_2}\cdots f_{\sigma_r}^{\gamma_r}$, where $\gamma_1, \gamma_2, \ldots, \gamma_r \in \{1, -1\}$. We shall prove by induction on ρ that

$$L(f_{\sigma_1}^{\gamma_1}f_{\sigma_2}^{\gamma_2}\cdots f_{\sigma_\rho}^{\gamma_\rho}) = \sum_{j=1}^{\rho} L(f_{\sigma_j}), \quad \rho = 1, 2, \ldots, r. \tag{47}$$

Indeed, the equation in (47) holds trivially for $\rho = 1$; and if the equation in (47) holds for some $\rho < r$, then

$$f_{\sigma_1}^{\gamma_1} f_{\sigma_2}^{\gamma_2} \cdots f_{\sigma_\rho}^{\gamma_\rho} = f_{\sigma_1}^{\gamma_1} \circ f_{\sigma_2}^{\gamma_2} \circ \cdots \circ f_{\sigma_\rho}^{\gamma_\rho}.$$

Since $w(f_\sigma)$ is freely reduced, we either have $\sigma_\rho \neq \sigma_{\rho+1}$, or $\sigma_\rho = \sigma_{\rho+1}$ and $\gamma_\rho = \gamma_{\rho+1}$. In the first case, we have

$$\varepsilon_0(f_{\sigma_\rho}^{\gamma_\rho}, f_{\sigma_{\rho+1}}^{\gamma_{\rho+1}}) = 0 = \varepsilon_0(f_{\sigma_1}^{\gamma_1} f_{\sigma_2}^{\gamma_2} \cdots f_{\sigma_\rho}^{\gamma_\rho}, f_{\sigma_{\rho+1}}^{\gamma_{\rho+1}})$$

by Lemma 24, so that the equation in (47) holds with ρ replaced by $\rho + 1$, while in the second case

$$\varepsilon_0(f_{\sigma_\rho}^{\gamma_\rho}, f_{\sigma_{\rho+1}}^{\gamma_{\rho+1}}) = \varepsilon_0(f_{\sigma_\rho}^{\gamma_\rho}, f_{\sigma_\rho}^{\gamma_\rho}) = 0 = \varepsilon_0(f_{\sigma_1}^{\gamma_1} f_{\sigma_2}^{\gamma_2} \cdots f_{\sigma_\rho}^{\gamma_\rho}, f_{\sigma_{\rho+1}}^{\gamma_{\rho+1}})$$

by Lemma 12, from which we conclude again that the equation in (47) holds with ρ replaced by $\rho+1$. This proves Assertion (47). It follows that a non-empty freely reduced word $w(f_\sigma)$ as above satisfies

$$L(w(f_\sigma)) = \sum_{j=1}^{r} L(f_{\sigma_j}) > 0;$$

in particular, $w(f_\sigma) \neq \mathbf{1}_G$. This shows that the subgroup $\mathfrak{F}_S$ is freely generated by the test functions f_σ, as claimed.

In order to establish (45), it clearly suffices to show that every subgroup of $\mathcal{RF}(G)$ generated by finitely many members of the family $\{f_\sigma\}_{\sigma\in S}$ meets $E(G)$ trivially. Let $\sigma_1, \sigma_2, \ldots, \sigma_k \in S$ be distinct indices, where $k \geq 1$. Then

$$\mathcal{N} := \langle f_{\sigma_1}, f_{\sigma_2}, \ldots, f_{\sigma_k} \rangle \cap E(G)$$

is normal in the finitely generated free group

$$\mathcal{F} := \langle f_{\sigma_1}, f_{\sigma_2}, \ldots, f_{\sigma_k} \rangle,$$

and the quotient

$$\mathcal{F}/\mathcal{N} \cong \mathcal{F}E(G)/E(G)$$

projects onto

$$\tilde{\mathcal{F}} := \langle \mathfrak{ab}_0(G)(f_{\sigma_1}), \mathfrak{ab}_0(G)(f_{\sigma_2}), \ldots, \mathfrak{ab}_0(G)(f_{\sigma_k}) \rangle \leq \overline{\mathcal{RF}_0(G)}.$$

By Part (ii), $\tilde{\mathcal{F}}$ if free abelian of rank k; in particular, $(\mathcal{F} : \mathcal{N}) = \infty$, implying $\mathcal{N} = 1$ by a result of Greenberg [5]; cf. also [7, Chap. I, Prop. 3.11]. Hence, Assertion (45) is proven. It follows that $\mathfrak{F}_S$ is hyperbolic, and that the restriction $\pi|_{\mathfrak{F}_S} : \mathfrak{F}_S \to \mathcal{RF}_0(G)$ is an embedding, so $\pi(\mathfrak{F}_S)$ is free with basis $\{\pi(f_\sigma)\}_{\sigma\in S}$ as claimed. The proof of Proposition 36 is complete. □

We are finally in a position to state and prove the following structure theorem concerning $\mathcal{RF}(G)$ and its quotient group $\mathcal{RF}_0(G)$.

Theorem 37 *Let G be a non-trivial group, set $\mathfrak{c}_G := |G|^{2^{\aleph_0}}$, and assume the axiom of choice. Then the following holds true.*

(i) *The groups* $\mathcal{RF}(G)$ *and* $\mathcal{RF}_0(G)$ *contain a free subgroup of rank* $\mathfrak{c}_G$, *but are not free*; *in particular*, $|\mathcal{RF}_0(G)| = \mathfrak{c}_G$.

(ii) *Every non-trivial torsion-free abelian group of rank at most* $2^{\aleph_0}$ *is realized* (*up to isomorphism*) *as the centralizer of a hyperbolic element in* $\mathcal{RF}(G)$.

(iii) *The abelianized groups* $\overline{\mathcal{RF}(G)}$ *and* $\overline{\mathcal{RF}_0(G)}$ *contain a* $\mathbb{Q}$-*vector space of dimension* $\mathfrak{c}_G$ *as a direct summand*; *in particular, these groups contain a free abelian subgroup of rank* $\mathfrak{c}_G$, *but are not free abelian*, *and*

$$\left|\overline{\mathcal{RF}(G)}\right| = \mathfrak{c}_G = \left|\overline{\mathcal{RF}_0(G)}\right|.$$

(iv) *Every non-trivial normal subgroup* $\mathcal{N} \trianglelefteq \mathcal{RF}(G)$ *contains a free subgroup of rank* $\mathfrak{c}_G$; *in particular*, $|\mathcal{N}| = \mathfrak{c}_G$ *and* $\mathcal{N}$ *is not soluble.*

(v) *If* $\mathcal{N} \trianglelefteq \mathcal{RF}(G)$ *has a non-trivial elliptic element*, *then* $\mathcal{N}$ *contains a subgroup isomorphic to a free power of the form* $U_0^{*\mathfrak{c}_G}$, *where* $U_0 := \mathcal{N} \cap G_0$.

Proof Let $\{f_\sigma\}_{\sigma\in S}$ be a family of pairwise locally incompatible test functions with $|S| = \mathfrak{c}_G$, and $C_{\mathcal{RF}(G)}(f_\sigma) \cong (\mathbb{Q}, +)$ for all $\sigma \in S$ (such a family exists by Theorem 30). By Part (i) of Proposition 36, the group $\mathfrak{F}_S = \langle f_\sigma : \sigma \in S\rangle$ is a free subgroup of $\mathcal{RF}(G)$ of rank $\mathfrak{c}_G$; and its image under the projection π is a free subgroup of $\mathcal{RF}_0(G)$ of the same rank. This last fact forces $|\mathcal{RF}_0(G)| \geq \mathfrak{c}_G$, and equality in this equation follows (in the presence of the axiom of choice) from Corollary 33.

Next, set

$$\mathcal{C} := \left\langle C_{\mathcal{RF}(G)}(f_\sigma) : \sigma \in S\right\rangle \leq \mathcal{RF}(G),$$

and consider the homomorphic images $\mathcal{C}' := \mathfrak{ab}(G)(\mathcal{C})$ and $\mathcal{C}_0' := \mathfrak{ab}_0(G)(\mathcal{C})$ of $\mathcal{C}$ in $\overline{\mathcal{RF}(G)}$ and $\overline{\mathcal{RF}_0(G)}$, respectively. Both $\mathcal{C}'$ and $\mathcal{C}_0'$ are divisible, and contain a free abelian subgroup of rank $\mathfrak{c}_G$ by Part (ii) of Proposition 36, namely the group $\mathfrak{ab}(G)(\mathfrak{F}_S)$ respectively $\mathfrak{ab}_0(G)(\mathfrak{F}_S)$. By the structure theorem on divisible groups, $\mathcal{C}'$ and $\mathcal{C}_0'$ decompose as $\mathcal{C}' = V' \oplus T'$ respectively $\mathcal{C}_0' = V_0' \oplus T_0'$, where V' and V_0' are $\mathbb{Q}$-vector spaces, and T', T_0' are the respective torsion subgroups of $\mathcal{C}'$ and $\mathcal{C}_0'$; cf. [4, Theorem 19.1]. Since $\mathfrak{ab}(G)(\mathfrak{F}_S)$ and $\mathfrak{ab}_0(G)(\mathfrak{F}_S)$ are torsion-free, they are embedded via the canonical projection $V' \oplus T' \to V'$ respectively $V_0' \oplus T_0' \to V_0'$ into V' respectively V_0', implying $\dim_{\mathbb{Q}} V' \geq \mathfrak{c}_G$ and $\dim_{\mathbb{Q}} V_0' \geq \mathfrak{c}_G$; and, since $|\mathcal{RF}(G)| = \mathfrak{c}_G$ by Corollary 33, we have

$$\dim_{\mathbb{Q}} V' = \mathfrak{c}_G = \dim_{\mathbb{Q}} V_0'.$$

Further, by a result of Baer, a divisible subgroup of an abelian group is a direct summand; cf. [1] or [4, Theorem 18.1]. It follows that V' is a direct summand of $\overline{\mathcal{RF}(G)}$, and that V_0' is a direct summand of $\overline{\mathcal{RF}_0(G)}$. The assertions concerning the cardinalities of $\overline{\mathcal{RF}(G)}$ and $\overline{\mathcal{RF}_0(G)}$ follow from the above plus Corollary 33.

Part (ii) of the theorem follows immediately from Proposition 7, Lemma 12, and Corollary 32.

The fact that $\mathcal{RF}(G)$ (for $G \neq \{1_G\}$) is not a free group, follows for instance from the existence of non-trivial elements with non-cyclic centralizer; or from the (already established) fact that $\overline{\mathcal{RF}(G)}$ is not free abelian. Similarly, the fact that $\overline{\mathcal{RF}_0(G)}$ is not free abelian serves to show that $\mathcal{RF}_0(G)$ itself is not free; alternatively, Corollary 32, in conjunction with Corollaries 21 and 22, allows us to exhibit non-trivial elements with non-cyclic centralizer in $\mathcal{RF}_0(G)$.

Next, we prove (iv). Let $\{f_\sigma\}_{\sigma\in S}$ be a family of test functions as described in Corollary 31 with $L(f_\sigma)=1$ for all $\sigma\in S$, say. Since $\mathcal{N}$ is non-trivial, it must (according to Proposition 2 and Corollary 3) contain a hyperbolic element h and, since $\mathcal{N}$ is normal, $h_1=c(h)\in\mathcal{N}$. By definition, h_1 is cyclically reduced, and $\alpha_1:=L(h_1)>0$ in view of Proposition 7, since h is hyperbolic. We claim that for all but at most two indices $\sigma\in S$ we have

$$f_\sigma h_1 f_\sigma^{-1} = f_\sigma\circ h_1\circ f_\sigma^{-1}. \tag{48}$$

Indeed, suppose there are three distinct indices $\sigma_1,\sigma_2,\sigma_3\in S$ such that

$$\varepsilon_0(f_{\sigma_i},h_1)+\varepsilon_0(h_1,f_{\sigma_i}^{-1})>0,\quad i=1,2,3.$$

Then there are two indices out of these three, to fix ideas say σ_1 and σ_2, such that

$$\varepsilon_0(f_{\sigma_1},h_1)>0\quad\text{and}\quad\varepsilon_0(f_{\sigma_2},h_1)>0$$

or

$$\varepsilon_0(f_{\sigma_1},h_1^{-1})>0\quad\text{and}\quad\varepsilon_0(f_{\sigma_2},h_1^{-1})>0.$$

In both cases, we conclude that there exists $\varepsilon>0$ such that

$$f_{\sigma_1}\left(1-\frac{\varepsilon}{2}+\eta'\right)=f_{\sigma_2}\left(1-\frac{\varepsilon}{2}+\eta'\right),\quad |\eta'|<\frac{\varepsilon}{2},$$

contradicting the fact that f_{σ_1} and f_{σ_2} are locally incompatible. Hence, at most two of the test functions f_σ exhibit cancellation when conjugating h_1. Deleting these exceptional functions, we obtain a family $\{f_\sigma\}_{\sigma\in S'}$ of pairwise locally incompatible test functions with $|S'|=|S|$, such that (48) holds for all $\sigma\in S'$.

We claim that the subgroup

$$\mathcal{F}:=\left\langle f_\sigma h_1 f_\sigma^{-1}:\sigma\in S'\right\rangle\le\mathcal{N}$$

is freely generated by the elements $f_\sigma h_1 f_\sigma^{-1}$ with $\sigma\in S'$. To see this, consider a reduced word

$$w=w(f_\sigma h_1 f_\sigma^{-1})=f_{\sigma_1}h_1^{\gamma_1}f_{\sigma_1}^{-1}f_{\sigma_2}h_1^{\gamma_2}f_{\sigma_2}^{-1}\cdots f_{\sigma_k}h_1^{\gamma_k}f_{\sigma_k}^{-1}, \tag{49}$$

where $k\ge 0$, $\gamma_j\in\mathbb{Z}\setminus\{0\}$, and $\sigma_1,\sigma_2,\dots,\sigma_k\in S'$ are indices such that $\sigma_j\ne\sigma_{j+1}$ for $j+1,2,\dots,k-1$. Since h_1 is cyclically reduced, we have

$$h_1^{\gamma_j}=\underbrace{h_1^{\operatorname{sgn}(\gamma_j)}\circ\cdots\circ h_1^{\operatorname{sgn}(\gamma_j)}}_{|\gamma_j|\text{ factors}},$$

and since, for $\sigma\in S'$,

$$\varepsilon_0(f_\sigma,h_1)=0=\varepsilon_0(h_1,f_\sigma^{-1})$$

by construction and $L(h_1)>0$, we have

$$f_{\sigma_j}h_1^{\gamma_j}f_{\sigma_j}^{-1}=f_{\sigma_j}\circ\underbrace{h_1^{\operatorname{sgn}(\gamma_j)}\circ\cdots\circ h_1^{\operatorname{sgn}(\gamma_j)}}_{|\gamma_j|\text{ factors}}\circ f_{\sigma_j}^{-1};$$

in particular,

$$L(f_{\sigma_j} h_1^{\gamma_j} f_{\sigma_j}^{-1}) = |\gamma_j|\alpha_1 + 2.$$

Moreover, by Part (i) of Lemma 24,

$$\varepsilon_0(f_{\sigma_j} h_1^{\gamma_j} f_{\sigma_j}^{-1}, f_{\sigma_{j+1}} h_1^{\gamma_{j+1}} f_{\sigma_{j+1}}^{-1}) = \varepsilon_0(f_{\sigma_j}^{-1}, f_{\sigma_{j+1}}) = 0,$$

from which we conclude that

$$L(w) = \alpha_1(|\gamma_1| + \cdots + |\gamma_k|) + 2k.$$

Hence, $w = \mathbf{1}_G$ forces $k = 0$, so that w has to be the empty word.

Finally, we establish Part (v). Since every non-trivial element of $f_\sigma G_0 f_\sigma^{-1}$ has length 2, we have

$$G_0 \neq f_\sigma G_0 f_\sigma^{-1}, \quad \sigma \in S;$$

and $f_{\sigma_1} G_0 f_{\sigma_1}^{-1} = f_{\sigma_2} G_0 f_{\sigma_2}^{-1}$ for $\sigma_1, \sigma_2 \in S$ implies $\sigma_1 = \sigma_2$, since f_{σ_1} and f_{σ_2} are locally incompatible for $\sigma_1 \neq \sigma_2$. Hence, the union

$$U_0 \cup \bigcup_{\sigma \in S} f_\sigma U_0 f_\sigma^{-1}$$

forms an amalgam with trivial intersection by Corollary 8. Now consider the canonical projection

$$\varphi : U_0 * \mathop{*}_{\sigma \in S} f_\sigma U_0 f_\sigma^{-1} \longrightarrow \Big\langle U_0 \cup \bigcup_{\sigma \in S} f_\sigma U_0 f_\sigma^{-1} \Big\rangle,$$

and let

$$t_1 u_1 t_1^{-1} \cdot t_2 u_2 t_2^{-1} \cdots t_r u_r t_r^{-1} = \mathbf{1}_G \tag{50}$$

be a normal form element in the kernel of φ; that is, $t_j \in \{\mathbf{1}_G\} \cup \{f_\sigma\}_{\sigma \in S}$, $u_j \in U_0 \setminus \{\mathbf{1}_G\}$, and $L(t_j^{-1} t_{j+1}) > 0$. We claim that

$$L(t_1 u_1 t_1^{-1} \cdot t_2 u_2 t_2^{-1} \cdots t_i u_i t_i^{-1}) = 2 \sum_{j=1}^{i} L(t_j), \quad i = 1, 2, \ldots, r, \tag{51}$$

which would imply that the left-hand side of (50) is the empty word, so that φ is an isomorphism. The proof of (51) is by induction on i; the case where $i = 1$ being obviously true, since $u_1 \neq \mathbf{1}_G$, so that $\varepsilon_0(t_1 u_1, t_1^{-1}) = 0$. Suppose then that (51) holds for some i with $1 \leq i < r$, and consider the normal form word

$$t_1 u_1 t_1^{-1} \cdot t_2 u_2 t_2^{-1} \cdots t_i u_i t_i^{-1} \cdot t_{i+1} u_{i+1} t_{i+1}^{-1}.$$

Now there are several cases.

(i) *We have* $t_i = \mathbf{1}_G$ *and* $i = 1$. Then $t_{i+1} = t_2 = f_\sigma$ for some $\sigma \in S$; and, since $L(f_\sigma) > 0$ and $u_2 \neq \mathbf{1}_G$, the equation

$$L\left(\prod_{j=1}^{i+1} t_j u_j t_j^{-1}\right) = 2 \sum_{j=1}^{i+1} L(t_j) \tag{52}$$

holds in this case.

(ii) *We have* $t_i = \mathbf{1}_G$ *and* $i \geq 2$. Then $t_{i-1} = f_{\sigma_1}$ and $t_{i+1} = f_{\sigma_2}$ with $\sigma_1, \sigma_2 \in S$. If $\sigma_1 \neq \sigma_2$, then (52) follows from Lemma 24 plus the fact that $u_{i+1} \neq \mathbf{1}_G$, while, for $\sigma_1 = \sigma_2$, the same conclusion follows since $u_i, u_{i+1} \neq \mathbf{1}_G$.

(iii) *We have* $t_i = f_\sigma$ *for some* $\sigma \in S$, *and* $t_{i+1} = \mathbf{1}_G$. In this case, (52) holds trivially.

(iv) *We have* $t_i = f_{\sigma_1}$ *and* $t_{i+1} = f_{\sigma_2}$ *with* $\sigma_1, \sigma_2 \in S$. In this case, we must have $\sigma_1 \neq \sigma_2$, and (52) follows again, this time from Lemma 24 plus the fact that $u_{i+1} \neq \mathbf{1}_G$.

This completes the proof by induction of (51).

We have shown that

$$\Big\langle U_0 \cup \bigcup_{\sigma \in S} f_\sigma U_0 f_\sigma^{-1} \Big\rangle \cong U_0 * \mathop{*}_{\sigma \in S} f_\sigma U_0 f_\sigma^{-1} \cong U_0^{*\mathfrak{c}_G},$$

and the group described on the left-hand side is clearly contained in $\mathcal{N}$, finishing the proof of Part (v), and of the theorem. □

References

1. Baer, R.: The subgroup of the elements of finite order of an abelian group. Ann. Math. **37**, 766–781 (1936)
2. Chiswell, I.M.: Introduction to Λ-Trees. World Scientific, Singapore (2001)
3. Chiswell, I.M., Müller, T.W.: A Class of Groups with Canonical ℝ-tree Action. Springer Lecture Notes in Mathematics (to appear)
4. Fuchs, L.: Abelian Groups. Pergamon, Oxford (1960)
5. Greenberg, L.: Discrete groups of motions. Can. J. Math. **12**, 414–425 (1960)
6. Halmos, P.R.: Measure Theory, 9th printing. van Nostrand, Princeton (1964)
7. Lyndon, R.C., Schupp, P.E.: Combinatorial Group Theory. Springer, Berlin (1977)
8. Sierpiński, W.: Cardinal and Ordinal Numbers. Monographs of the Polish Academy of Science, vol. 34. Warsaw (1958)