ON ROMANOV'S CONSTANT

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Abstract. We show that the lower density of integers representable as a sum of a prime and a power of two is at least 0.107. We also prove that the set of integers with exactly one representation of the form $p+2^k$ has positive density. Previous results of this kind needed "at most 15" in place of "exactly one". To achieve this result we introduce a new method. In particular we make use of uneven distribution of sums of a power of two and a reduced residue class.

1. Introduction

1.1. **Summary of previous results.** De Polignac [12] stated that every odd number can be written as a sum of a prime and a power of 2. Soon he realized that this is not correct. (The counter example 959 already appeared in a letter by Euler to Goldbach, 16.12.1752, see [4]). Romanov [14] showed that a positive proportion of all integers can be written as the sum of a prime number and a power of two. Van der Corput [15] and Erdős independently showed that a positive proportion of all integers are not of the form $p + 2^k$. Erdős [3] actually constructed an arithmetic progression no member of which is of this form. For this purpose he introduced the concept of covering congruences.

There have been numerical investigations on the density which we briefly summarize. Let $d(N) := \frac{|\{n \le N : n = p+2^k,\ p \text{ prime}\}|}{N}$, and

$$\underline{d} = \liminf_{n\to\infty} d(N), \qquad \overline{d} = \limsup_{n\to\infty} d(N).$$

Y.G. Chen and X.G. Sun [1] proved $\underline{d} > 0.0869$. G. Lü [10] proved $\underline{d} > 0.09322$. Habsieger and Roblot [5] proved $\underline{d} > 0.0933$. Pintz [11] showed that $\underline{d} > 0.09367$, but made use of a result of Dong Wu, which according to [6] is incomplete and correcting this would only give $\underline{d} > 0.093626$. Habsieger and Sivak-Fischler [6] slightly improved upon this and hold the current record with $\underline{d} > 0.0936275$. Even though this last improvement appears to be small it provided a new more explicit variant of the Bombieri-Friedlander-Iwaniec type, and from this achieves some improvement over the best upper bound sieve results on twin primes. It appears that current improvements on this sieve bound for twin primes are rather involved but numerically tiny. Moreover Pintz [11] discusses that the bound above is "near to the best what can be expected at the present state of number theory along the lines of Romanov's idea". The discussion below of course uses the ideas of Romanov and others but the reason why we achieve an improvement of 17 percent is a change of strategy.

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2010 *Mathematics Subject Classification.* 11P32 Goldbach-type theorems; other additive questions involving primes.

Theorem 1. *For the lower density $\underline{d}$ of integers representable as a sum of a prime and a power of two the following estimate holds:*

$$\underline{d} \geq 0.107648.$$

Habsieger and Roblot [5] proved an upper bound of $\overline{d} \leq 0.49095$.

A heuristic study by Romani [13], based on a model suggested by Bombieri, suggested that the correct density could possibly be $d = 0.434...$

X.G. Sun [16] characterized the arithmetic progressions for which a positive proportion of their members are of the form $p + 2^k$. In particular he solved a problem by Erdős, by proving that arithmetic progressions without integers of the form $p + 2^k$ necessarily come from covering congruences.

Y.G. Chen and X.G. Sun [1] also studied the problem of the number of representations. Let A_k denote the set of positive integers n which have at most k distinct representations of the form $n = p + 2^k$. They proved that A_{16} has positive density. Pintz [11] proved this also for A_{15}. Assuming two quite hopeless conjectures, namely Romani's heuristic [13] that $d = 0.434...$, and an improvement of the upper sieve for twin primes, improving the current 3.910425 by Wu [17] to the conjectured $C = 1$, Pintz [11] noted that A_3 has positive lower density.

Theorem 2. *In the residue class* $1253689547594657608 \pmod{2^{240} - 1}$ *the set of integers representable as the sum of a prime and a power of two has relative density between* 0.01702 *and* 0.01848*. The set of integers in this residue class which has a unique representation as a sum of a prime and a power of two has relative density* ≥ 0.01557*. In particular, A_1 has positive density.*

Theorem 3. *For each k a positive proportion of all integers have at least k different representations.*

1.2. **Explanation of the new method.** We define $r(n)$ to be the number of representations of n as the sum of a prime and a power of two. Erdős [3] proved that for infinitely many integers $r(n) \gg \log\log n$ holds, and he conjectured that $r(n) = o(\log n)$ holds, commenting: “This if true is probably rather deep.” For the problem under consideration it is important to study when $r(n)$ is small.

Romanov's argument uses the Cauchy-Schwarz inequality in the following way: From the prime number theorem we have $S_1 = \sum_{n \leq x} r(n) \sim \frac{x}{\log 2}$, while we have

$$S_2 = \sum_{n \leq x} r(n)^2 = \#\{(p_1, p_2, a, b) : p_1 + 2^a = p_2 + 2^b \leq x\},$$

and by a sieve argument it is shown that the quantity on the right is

$$S_2 \leq Cx, \tag{1}$$

for some constant C depending on the quality of the sieve. Now the Cauchy-Schwarz inequality gives

$$\left(\sum_{n \leq x} r(n)\right)^2 \leq \left(\sum_{\substack{n \leq x \\ r(n) > 0}} 1\right) \left(\sum_{n \leq x} r(n)^2\right),$$

and we obtain $\#\{n \leq x : r(n) > 0\} \geq \frac{x}{C \log^2 2}$. There has been some work on the numerical value of this bound, but the strategy has not changed. Improvements for Romanov's constant have relied on improvements for the upper bound sieve or better bounds for the mean value of the singular series of the problem $p+2^a$. As shown above the numerical improvements over the last years have been small.

Here we change the strategy. In particular we improve the way how the Cauchy-Schwarz inequality is applied. The Cauchy-Schwarz inequality in the form $\langle v,w\rangle^2 \leq \|v\|\cdot\|w\|$ is strict if and only if the vectors v and w are collinear. In our application this means that $r(n)$ takes on only the values 0 and $C\log 2$. If we can find a large set of integers, on which $r(n)$ deviates from these two values, we obtain that the vectors are not collinear, and we obtain some improvement. Pintz [11] observed that since $C\log 2$ is not an integer, $r(n)$ cannot take on the value $C\log 2$ at all. However, the resulting improvement is rather small. In this article we use the fact that $r(n)$ is not equally distributed among the congruence classes to give a considerably larger improvement. Consider e.g. congruence classes modulo 3. Both, primes and powers of 2, are uniformly distributed on the classes 1 and 2 (mod 3). Therefore the sum of a prime and a power of 2 is divisible by 3 with probability $\frac{1}{2}$, and with probability $\frac{1}{4}$ in each of the other two residue classes. For this reason we expect that $r(n)$ typically takes on bigger values when $n\equiv 0 \pmod 3$, compared to the other classes $n\equiv 1,2 \pmod 3$.

We now compute lower bounds for the density in each residue class separately, and obtain an improved lower bound for Romanov's constant. For Theorem 1 we use the modulus $2^{24}-1$ instead of 3.

A crucial point in the argument is a sieve bound for prime twins with given distance. In our approach we need bounds for the number of prime pairs in a congruence class to a fixed modulus. The next result is Theorem 3.12 of Halberstam and Richert [7]. A theorem of this type goes back to Klimov [9].

Theorem 4. *For every fixed integer ℓ and every $\epsilon>0$ there exists some $x_0=x_0(\epsilon)$, such that for $x>x_0$ and $(k,\ell)=1$ we have that for every integer d the number of primes $p\leq x$, $p\equiv k \pmod{\ell}$, such that $p+d$ is prime, is bounded above by*

$$\frac{C_1 x}{\varphi(\ell)\log^2 x}\prod_{p>2}\Big(1-\frac{1}{(p-1)^2}\Big)\prod_{\substack{p\mid \ell d\\ p>2}}\Big(1+\frac{1}{p-2}\Big),$$

where $C_1=8+\epsilon$.

Remark 1. *Jie Wu informed us that the same result holds true with $C_1=7.8209$, the proof being an adaptation of his result on prime pairs without a congruence restriction (confer [17]). However, since the latter paper is rather complicated, and since such an adaptation is unlikely to appear in print in the near future, we decided to use only the value $C_1=8+\epsilon$. If we would use $C_1=7.8209$ instead, the density in Theorem 1 would improve to $\underline{d}\geq 0.110114$. It may be conjectured that Theorem 4 holds with $C_1=2+\epsilon$. Using this constant the density in Theorem 1 improves to $\underline{d}\geq 0.3458$.*

2. Numerical evaluation of certain sums

For an odd integer n let $\epsilon(n)$ denote the multiplicative order of 2 modulo n, i.e. the least positive integer satisfying $2^{\epsilon(n)}\equiv 1 \pmod n$. An essential point in the proof of Romanov's theorem is the convergence of the series $\sum_q \frac{1}{q\epsilon(q)}$. This step was greatly simplified by Erdős and Turán [2]. Giving a good numerical approximation to this sum is a more demanding task that was only recently satisfactorily solved

t	1	2	3	4	6	8	12	24
$S(t) \geq$	1.01583	1.04481	1.02466	1.06264	1.07819	1.06547	1.11641	1.12861
$S(t) \geq$	1.01598	1.04536	1.02514	1.06445	1.08095	1.06757	1.12415	1.13875
Σ_1	1.20940	1.71202	1.43547	2.48899	2.73396	2.62430	4.71372	5.54272
Σ_2	81482	403205	211290	851695	2501246	1056255	5182153	6370500
$S(t) \leq$	1.01609	1.04568	1.02545	1.06517	1.08269	1.06864	1.12771	1.14370

TABLE 1. First line gives the lower bound for $S(t)$ by considering all integers in the sum with $\epsilon(n) \leq 300$. The second line gives the lower bound by additionally including all n with $\epsilon(n) \leq 2 \cdot 10^5$ and $P^+(n) \leq 10^8$. Σ_1 and Σ_2 are the sums occurring in Lemma 2, the latter restricted to integers m with $P^+(m) \leq 10^8$, and the last line gives the upper bound thus obtained. Lower bounds are rounded down, upper bounds are rounded up.

by Khalfalah and Pintz [8]. For our approach we need a somewhat refined version of this sum. Define the multiplicative function $f(n)$ by $f(p) = \frac{1}{p-2}$ for p an odd prime, and $f(q) = 0$ if q is 2 or a proper power of a prime. Let t be a divisor of 24. Then define the sum

$$S(t) = \sum_{\substack{d:(\epsilon(d),24)|t\\(d,2^{25}-2)=1}} \frac{f(d)(\epsilon(d),t)}{\epsilon(d)}.$$

The aim of this section is to describe the computation leading to the following.

Theorem 5. *The function $S(t)$ is bounded above by the values given in the last line of Table 1.*

Clearly the difficulty in computing $S(t)$ lies in factoring $2^m - 1$. Once this can be done for all m up to some bound D, say, the remainder of the sum can be estimated using the following result by Chen and Sun [1].

Lemma 1. *We have*

$$\sum_{\epsilon(n)>D} \frac{f(n)}{\epsilon(n)} \leq 2.7961 \frac{\log D}{D}.$$

Using only this result computing $S(24)$ with an error ≤ 0.1 would involve the factorization of $2^m - 1$ for all m up to almost 6000, which is clearly impossible.[1] However, it is not too difficult to compute all *small* prime factors of $2^m - 1$ for much larger values of m. Hence we decompose $S(t)$ according to

$$\sum_{(d,24)|t} \sum_{\substack{\epsilon(n)=d\\(n,2^{25}-2)=1}} = \sum_{d<D_1} + \sum_{\substack{D_1 \leq d \leq D_2\\P^+(n)\leq P}} + \sum_{\substack{D_1 \leq d \leq D_2\\P^+(n)>P}} + \sum_{d>D_2}, \tag{2}$$

where $P^+(n)$ denotes the largest prime factor of n, and on the right hand side only the additional summation conditions are given.

[1]The Cunningham project http://homes.cerias.purdue.edu/~ssw/cun/, see also http://www.mersennewiki.org/index.php/2_Minus_Tables informs about the state of art of factorization of integers of the form $2^n - 1$. At the time of writing, for example, the prime factors of $2^{929} - 1$ are not yet known.

The first two of these sums will be computed explicitly. Since for a multiplicative function f the sum $\sum_{\epsilon(d)|e} f(d)$ is much easier to compute than $\sum_{\epsilon(d)=e} f(d)$, we computed the former sum for all $e \leq 300$, and deduced from this the values of the latter sum via the following Möbius inversion identity:

$$\sum_{\epsilon(d)=e} h(d) = \sum_{t|e} \mu(\frac{e}{t}) \sum_{\epsilon(d)|e} h(d),$$

applied with $h = f$. For the last sum of equation (2) we use the bound given by Lemma 1. The estimation of the third sum is the content of the following.

Lemma 2. *With integers t, D_1, D_2, P we have*

$$\sum_{\substack{D_1 \leq \epsilon(n) \leq D_2 \\ P^+(n) > P \\ (n, 2^{24-1})=1 \\ (\epsilon(n), 24)|t}} \frac{f(n)(\epsilon(n), 24)}{\epsilon(n)} \leq \frac{D_1 \log 2}{2(P-2)\log P} \sum\nolimits_1 + \frac{1}{(P-2)\log P} \sum\nolimits_2$$
$$+2.7961 \frac{t \log^2 D_2}{2(P-2)\log P} + 2.7961 \frac{t D_1 \log 2 \log D_1}{2(P-2)\log P},$$

where

$$\sum\nolimits_1 = \sum_{\substack{\epsilon(n) \leq D_1 \\ (n, 2^{24-1})=1 \\ (\epsilon(n), 24)|t}} f(n)t, \qquad \sum\nolimits_2 = \sum_{\substack{D_1 \leq d \leq D_2 \\ (d, 24)|t}} \frac{\varphi(d)}{d} \sum_{\substack{\epsilon(n) \leq d \\ (n, 2^{24-1})=1 \\ (\epsilon(n), 24)|t}} f(n)([d, \epsilon(n)], 24).$$

Proof. Fix a prime number p, and define $g(n)$ as $g(n) = ([\epsilon(n), \epsilon(p)], 24)$, $h(n) = [\epsilon(n), \epsilon(p)]$, and $\mu = \max(D_1, \epsilon(p))$. The contribution of integers divisible by p equals

$$\begin{aligned}
\sum_{\substack{D_1 \leq \epsilon(n) \leq D_2 \\ p|n \\ (n, 2^{24-1})=1 \\ (\epsilon(n), 24)|t}} \frac{f(n)(\epsilon(n), 24)}{\epsilon(n)} &= \frac{1}{p-2} \sum_{\substack{D_1 \leq h(n) \leq D_2 \\ p \nmid n \\ (n, 2^{24-1})=1 \\ g(n)|t}} \frac{f(n)g(n)}{[\epsilon(n), \epsilon(p)]} \\
&\leq \frac{1}{p-2} \sum_{\substack{D_1 \leq h(n) \leq D_2 \\ (n, 2^{24}-1)=1 \\ g(n)|t}} \frac{f(n)g(n)}{\max(\mu, \epsilon(n))} \\
&\leq \frac{1}{p-2} \Big(\frac{1}{\mu} \sum_{\substack{D_1 \leq h(n) \leq D_2 \\ \epsilon(n) \leq \mu \\ (n, 2^{24-1})=1 \\ g(n)|t}} (f(n)g(n) + 2.7961 t \frac{\log \mu}{\mu} \Big).
\end{aligned}$$

The number of primes $p > P$ which divide some $2^d - 1$ is at most $\frac{\log(2^d-1)}{\log P} \leq \frac{d \log 2}{\log P}$. We first consider primes p with $\epsilon(p) < D_1$. Then $\mu = D_1$, the number of such primes is $\leq \sum_{d < D_1} \frac{d \log 2}{\log P} < \frac{D_1^2 \log 2}{2 \log P}$. Hence this part of the sum gives a contribution bounded above by

$$\frac{D_1 \log 2}{2(P-2)\log P}\Big(\sum_{\substack{\epsilon(n)\leq D_1\\ (n,2^{24-1})=1\\ g(m)|t}} f(n)g(n) + 2.7961t\log D_1\Big).$$

Next consider a prime with $\epsilon(p) = d$ with $D_1 \leq d \leq D_2$. There are at most $\frac{\varphi(d)\log 2}{\log P}$ such primes, hence the contribution of primes with a fixed value d is at most

$$\frac{\varphi(d)}{d(P-2)\log P}\Big(\sum_{\substack{\epsilon(n)\leq d\\ (n,2^{24-1})=1\\ g(n)|t}} f(n)g(n) + 2.7961t\log d\Big).$$

We have

$$\sum_{D_1\leq d\leq D_2} \frac{\log d}{d} \leq \int\limits_{D_1-1}^{D_2} \frac{\log t}{t}\,dt = \frac{1}{2}\big(\log^2 D_2 - \log^2(D_1-1)\big).$$

Putting these bounds together we obtain our claim. □

We computed the complete factorization of $2^d - 1$ for $d \leq 300$, and determined all prime factors $p \leq 10^8$ of $2^d - 1$ for $d \leq 2\cdot 10^5$. Note that the factorization of $2^d - 1$ for $d \leq 300$ also gives the precise value of the first sum occurring in Lemma 2, and the partial factorization for $301 \leq d \leq 2\cdot 10^5$ yields the contribution of integers m having no prime factor $\leq 10^8$ to the second sum. Moreover, we have

$$\sum_{m|N} f(m) \leq \Big(1+\frac{1}{P-2}\Big)^{\frac{\log N}{\log P}} \sum_{\substack{m|N\\ P^+(m)\leq P}} f(m) \leq e^{\frac{\log N}{(P-2)\log P}} \sum_{\substack{m|N\\ P^+(m)\leq P}} f(m),$$

thus

$$\sum_{\substack{\epsilon(m)=d\\ (m,2^{24-1})=1}} f(m) \leq 2^{\frac{d}{(P-2)\log P}} \sum_{\substack{\epsilon(m)=d\\ (m,2^{24-1})=1\\ P^+(m)\leq P}} f(m).$$

Taking $D = 2\cdot 10^5$ and $P = 10^8$ we see that the exponential factor is at most $1 + 1.16\cdot 10^{-5}$, thus we obtain a quite precise upper bound for the second sum in Lemma 2 as well.

Putting $D_1 = 301$, $D_2 = 10000$, $P = 10^8$ we obtain the lower and upper bounds for $S(t)$ given in Table 1.

3. Proof of the main results

Define for integers k, ℓ with k odd

$$S_1(x,k,\ell) = \sum_{\substack{n\leq x\\ n\equiv k\ (\ell)}} r(n),$$

$$S_2(x,k,\ell) = \sum_{\substack{n\leq x\\ n\equiv k\ (\ell)}} r(n)^2.$$

Then we have

$$\begin{aligned}S_2(x,k,\ell) &= \#\{p_1+2^{a_1}=p_2+2^{a_2}\equiv k\ (\ell)\}\\ &= \sum_{\substack{\kappa\le\ell,\alpha\le\epsilon(\ell)\\ \kappa+2^\alpha\equiv k\ (\ell)}} \#\{p+2^a\equiv k\ (\ell), p\equiv\kappa\ (\ell)\}\\ &\quad + \sum_{\substack{\kappa_1,\kappa_2\le\ell,\alpha_1,\alpha_2\le\epsilon(\ell)\\ \kappa_i+2^{\alpha_i}\equiv k\ (\ell)}} \#\{p_1+2^{a_1}=p_2+2^{a_2}, p_1\neq p_2, p_i\equiv\kappa_i\ (\ell), a_i\equiv\alpha_i\ (\epsilon(\ell))\}.\end{aligned}$$

Now fix $a_1, a_2, \kappa_1, \kappa_2$, such that $\kappa_1+2^{a_1}\equiv\kappa_2+2^{a_2} \pmod{\ell}$. Then we estimate the number $N(x; a_1, a_2, \kappa_1, \kappa_2)$ of primes p_1, p_2 with $p_i\equiv\kappa_i \pmod{\ell}$ and $p_1-p_2=2^{a_1}-2^{a_2}$ using an upper bound sieve.

More precisely we apply Theorem 4 and obtain for ℓ fixed and x sufficiently large

$$N(x; a_1, a_2, \kappa_1, \kappa_2) \le \frac{C_1C_2x}{\varphi(\ell)\log^2 x}\prod_{\substack{p\mid\ell(2^a-2^b)\\ p>2}}\Big(1+\frac{1}{p-2}\Big)$$

where

$$C_2=\prod_{p>2}\Big(1-\frac{1}{(p-1)^2}\Big)=0.6601\ldots,$$

and $C_1 = 8+\epsilon$ (resp. $C_1 = 7.8209$, see Remark 1). We now fix $\ell = 2^m-1$. Put $L=\frac{x}{\log 2}$. We have

$$\begin{aligned}\sum_{\substack{a_1<a_2\le L\\ a_i\equiv\alpha_i\ (m)}}\prod_{p\mid\ell(2^{a_1}-2^{a_2})}\Big(1+\frac{1}{p-2}\Big) &= \prod_{p\mid\ell}\Big(1+\frac{1}{p-2}\Big)\sum_{\substack{a_1<a_2\le L\\ a_i\equiv\alpha_i\ (m)}}\sum_{\substack{(d,2\ell)=1\\ d\mid 2^{a_1}-2^{a_2}}} f(d)\\ &\sim \frac{1}{2m^2}\prod_{p\mid\ell}\Big(1+\frac{1}{p-2}\Big)L^2\sum_{\substack{(d,2\ell)=1\\ (\epsilon(d),m)\mid(\alpha_1-\alpha_2,m)}}\frac{f(d)(\epsilon(d),m)}{\epsilon(d)}\\ &= \frac{C_3L^2}{2m^2}S\big((\alpha_1-\alpha_2,m)\big), \qquad (3)\end{aligned}$$

where $C_3=\prod_{p\mid\ell}\big(1+(p-2)^{-1}\big)$.

Then summing over all possible choices for $\kappa_1, \kappa_2, \alpha_1, \alpha_2$ we obtain

$$\begin{aligned}S_1(x,k,\ell) &\sim \frac{x}{\varphi(\ell)m\log 2}\#\{\kappa,\alpha : (\kappa,\ell)=1, \kappa+2^\alpha\equiv k\ (\ell)\}\\ S_2(x,k,\ell) &\le \frac{x}{\varphi(\ell)m\log 2}\#\{\kappa,\alpha : (\kappa,\ell)=1, \kappa+2^\alpha\equiv k\ (\ell)\}\\ &\quad +\frac{C_1C_2C_3x}{\varphi(\ell)m^2\log^2 2}\sum_{\kappa_1+2^{\alpha_1}\equiv\kappa_2+2^{\alpha_2}\equiv k\ (\ell)} S\big((\alpha_1-\alpha_2,m)\big)\end{aligned}$$

We now estimate the density of integers n with $r(n)>0$ using the following lemma due to Pintz [11, Lemma 4'], which improves on the use of the Cauchy-Schwarz-inequality exploiting the fact that $r(n)$ takes on integral values only.

Lemma 3. *Suppose that $b(n)\in\mathbb{N}$ for each $n\in N$. Assume that*

$$\sum_{n=1}^N b(n)=M, \qquad \sum_{n=1}^N b(n)^2=DM.$$

Then

$$\#\{n \leq N : b(n) > 0\} \geq \frac{\lceil D \rceil + \lfloor D \rfloor - D}{\lceil D \rceil \lfloor D \rfloor} M.$$

For each k we let $\delta(k, \ell)$ be the lower bound for the relative density of integers with $r(n) > 0$ within the residue class $k \pmod{\ell}$ obtained by applying the previous lemma. Then computing the average of $\delta(k, \ell)$ for all k gives a lower bound for the density of all $r(n) > 0$.

For the numerical computation we now put $m = 24$, i.e. $\ell = 16777215 = 9 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241$. Then we get $C_3 = 3.73922$, and the upper bounds for $S(t)$ as computed in the previous section. For each $k \leq \ell$ we now compute $\delta(k, \ell)$. There are residue classes modulo $2^{24} - 1$ which do not contain integers representable as the sum of a prime and a power of 2. Discarding these residue classes we find that $\delta(k, \ell)$ varies between 0.06693 and 0.11838. Summing up these values we obtain that the total density of integers representable as the sum of a prime and a power of two is at least 0.10788. It is striking how much the distribution of $\delta(k, \ell)$ is skewed: The maximum is just 10% above its mean value, while deviation of the small values from the mean is considerably larger.

To prove Theorem 3 note that the residue classes -2^i, $0 \leq i \leq m$ are coprime to $\ell = 2^m - 1$, hence

$$S_1(x, 0, \ell) \sim \frac{x}{\varphi(\ell) \log 2},$$

that is, the average number of representations of an integer in the residue class $0 \pmod{\ell}$ is $\frac{\ell}{\varphi(\ell) \log 2}$. If we take $m = k!$, then $2^m - 1$ is divisible by all prime numbers up to k, thus $\frac{\ell}{\varphi(\ell)} \geq \frac{k!}{\varphi(k!)} \geq \log k$.

Denote by $N(x)$ the number of integers $n \leq x$ satisfying $n \equiv 0 \pmod{\ell}$ and $r(n) > \frac{\ell}{2\varphi(\ell) \log 2}$. Then we have

$$\sum_{n \leq x} r^2(n) \geq \sum_{\substack{n \leq x \\ n \equiv 0 \pmod{\ell} \\ r(n) > \frac{\ell}{2\varphi(\ell)}}} r^2(n) \geq \frac{1}{N(x)} \left(\sum_{\substack{n \leq x \\ n \equiv 0 \pmod{\ell} \\ r(n) > \frac{\ell}{2\varphi(\ell)}}} r(n) \right)^2 \geq \frac{x^2}{4N(x)\varphi(\ell)^2 \log^2 2}.$$

With equation (1) we conclude for k sufficiently large

$$N(x) \geq \frac{x}{4C\varphi(\ell)^2 \log^2 2} \geq \frac{x \log^2 k}{4C\ell^2 \log^2 2} \geq 2^{-2(k!)} x.$$

This proves Theorem 3.

Lemma 4. *The residue class* $a = 1253689547594657608 \pmod{2^{240} - 1}$ *has a unique representation as a sum of a power of 2 and an integer coprime to* $2^{240} - 1$.

The verification of this lemma can easily be done by a computer calculation. However, it seems more interesting to explain how to obtain the residue class in question. Let us start from the residue class $a_1 = 253 \pmod{2^{24} - 1}$, which has the unique representation $253 \equiv 251 + 2^1$ as the sum of a power of 2 and an integer coprime to $2^{24} - 1$. This residue class was found during the computations necessary for Theorem 1, where we computed the number of representations of each class modulo $2^{24} - 1$ individually. Suppose that a_2 is a residue class modulo $2^{48} - 1$, which is congruent to a_1 modulo $2^{24}-1$. Then a_2 has at most two representations as the sum of a power of two and an integer coprime to $2^{48}-1$, namely $(a_2-2)+2^1$ and $(a_2-2^{25})+2^{25}$. The prime number 97 divides $2^{48}-1$, but not $2^{24}-1$. We now determine $x \pmod{97}$ in such a way that $253+x(2^{24}-1)-2^{25}$ is divisible by 97, and obtain a residue class a_2, which has at most one representation. Further, we pass to a residue class a_3 modulo $2^{240}-1$, such that $a_3 \equiv a_2 \pmod{2^{48}-1}$. Then a_3 has at most 5 representations, and these representations are $(a_2-2)+2^{1+48i}$, $i=0,1,\dots,4$. The prime numbers 11, 31, 41, 61 divide $2^{240}-1$, but not $2^{48}-1$. We now determine $y \pmod{11\cdot 31\cdot 41\cdot 61}$ such that

$$\begin{aligned}
a_2-y(2^{48}-1)-2^{49} &\equiv 0 \pmod{11}\\
a_2-y(2^{48}-1)-2^{97} &\equiv 0 \pmod{31}\\
a_2-y(2^{48}-1)-2^{145} &\equiv 0 \pmod{41}\\
a_2-y(2^{48}-1)-2^{193} &\equiv 0 \pmod{61}
\end{aligned}$$

In this way we obtain the residue class $a=1253689547594657608$, which has at most one representation. We directly check that $(1253689547594657608-2, 2^{240}-1)=1$, and find that this residue class has exactly one representation.

For the residue class $a \pmod{2^{240}-1}$ we obtain

$$\begin{aligned}
S_1(x,a,2^{240}-1) &\sim \frac{x}{\varphi(2^{240}-1)240\log 2}\\
S_2(x,a,2^{240}-1) &\leq \frac{x}{\varphi(2^{240}-1)240\log 2}\\
&\quad+\frac{C_1C_2x}{\varphi(2^{240}-1)240^2\log^2 2}\prod_{p|2^{240}-1}\left(1+(p-2)^{-1}\right)\sum_{(d,2^{241}-2)=1}\frac{f(d)(\epsilon(d),240)}{\epsilon(d)}\\
&= \frac{x}{\varphi(2^{240}-1)240\log 2}\Big(1+0.1429\sum_{(d,2^{241}-2)=1}\frac{f(d)(\epsilon(d),240)}{\epsilon(d)}\Big).
\end{aligned}$$

We now compute the sum on the right as described in section 2 to be ≤ 1.0991. We therefore obtain

$$S_2(x,a,2^{240}-1)\leq 1.15705\frac{x}{\varphi(2^{240}-1)240\log 2}.$$

Let $N_j(x)$ be the number of integers $n\leq x$ with $n\equiv a \pmod{2^{240}-1}$ with $r(n)=j$. Then we have

$$\begin{aligned}
S_1(x,a,2^{240}-1) &= \textstyle\sum_{j\geq 1} jN_j(x) &&\sim \tfrac{x}{\varphi(2^{240}-1)240\log 2}\\
S_2(x,a,2^{240}-1) &= \textstyle\sum_{j\geq 1} j^2N_j(x) &&\leq (1+c)\tfrac{x}{\varphi(2^{240}-1)240\log 2},
\end{aligned}$$

where we put $c=0.15705$. Multiplying the first equation by 2, and substracting the second we obtain

$$N_1(x)-\sum_{j\geq 3} j(j-2)N_j(x)\geq \frac{(1-c-o(1))x}{\varphi(2^{240}-1)240\log 2}.$$

Since the sum is non-negative, the relative density of integers $n\equiv a \pmod{2^{240}-1}$, which can uniquely be written as the sum of a prime and a power of two, is at least

$$\frac{(1-c)(2^{240}-1)}{240\varphi(2^{240}-1)\log 2}=0.01557\dots.$$

It follows from Lemma 3 with $D = 1 + c$ that in this residue class the relative density of integers which have at least one representation is at least 0.01702. On the other hand the density of such integers is at most

$$\frac{2^{240} - 1}{240\varphi(2^{240} - 1)\log 2} = 0.01848\ldots,$$

and Theorem 2 follows.

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