

# On Shanks' Algorithm for Modular Square Roots\*

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## Abstract

Let  $p$  be a prime number,  $p = 2^n q + 1$ , where  $q$  is odd. D. Shanks described an algorithm to compute square roots  $\pmod{p}$  which needs  $O(\log q + n^2)$  modular multiplications. In this note we describe two modifications of this algorithm. The first needs only  $O(\log q + n^{3/2})$  modular multiplications, while the second is a parallel algorithm which needs  $n$  processors and takes  $O(\log q + n)$  time.

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D. Shanks[3] gave an efficient algorithm for computing square roots modulo a prime. If  $p = 2^n q + 1$ , this algorithm consists of an initialization, which takes  $O(\log q)$  modular multiplications, and a loop, which is performed at most  $n$  times and needs  $n$  modular multiplications at most. Hence the total cost is  $O(\log q + n^2)$  modular multiplications. This is actually the normal running time, for S. Lindhurst [2] has shown that on average the loop needs  $\frac{1}{4}(n^2 + 7n - 12) + 1/2^{n-1}$  modular multiplications. For most prime numbers  $p$ ,  $n$  is much smaller than  $\sqrt{\log q}$ , hence the initialization will be the most costly part, however, prime numbers occurring "in practice" are not necessarily random, and if  $p - 1$  is divisible by a large power of 2, the loop becomes more expensive than the initialization. In this note we will give two modifications of Shanks' algorithm. The first algorithm needs only  $O(\log q + n^{3/2})$  modular multiplications, while the second is a parallel algorithm running on  $n$  processors which needs  $O(\log q + n)$  time. Here and in the sequel time is measured in modular multiplications, and we assume that all other operations are at most as expensive as a modular multiplication. On the

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other hand both our algorithms have larger space requirements. Whereas Shanks' algorithm has to store only a bounded number of residues  $\pmod{p}$ , our algorithms have to create two arrays, each containing  $n$  residues  $\pmod{p}$ . However, on current hardware this amount of memory appears easily manageable compared to the expenses of the computation.

D. Bernstein[1] found a different improvement of Shanks algorithm, which has running time  $O(\log q + n^2/\log^2 n)$ . Although asymptotically worse than our algorithm, it appears that Bernstein's algorithm is more efficient for medium sized values of  $n$ .

We assume that looking up an element in a table of length  $n$  is at most as expensive as a modular multiplication, an assumption which is certainly satisfied on any reasonable computer.

First we give a description of Shanks' algorithm. We assume that we are given a prime  $p = 2^n q + 1$ , a quadratic residue  $a$  and a non-residue  $u$ , and are to compute an  $x$  such that  $x^2 \equiv a \pmod{p}$ . Then the algorithm runs as follows.

**Algorithm 1:**

1. Set  $k = n$ ,  $z = u^q$ ,  $x = a^{(q+1)/2}$ ,  $b = a^q$ .
2. Let  $m$  be the least integer with  $b^{2^m} \equiv 1 \pmod{p}$ .
3. Set  $t = z^{2^k - m - 1}$ ,  $z = t^2$ ,  $b = bz$ ,  $x = xt$ .
4. If  $b = 1$ , stop and return  $x$ , otherwise set  $k = m$  and go to step 2.

It is easy to see that the congruence  $x^2 \equiv ab \pmod{p}$  holds at every stage of the algorithm, hence, if it terminates we really obtain a square root of  $a$ .

To show that this algorithm terminates after at most  $n$  loops, consider the order of  $b$  and  $z \pmod{p}$ . After the first step, the latter is  $2^n = 2^k$ , since  $u$  is a nonresidue, whereas the first one is strictly smaller, since  $a$  is a quadratic residue. In the second step the order of  $b$  is determined to be exactly  $2^m$ , and in the third step  $z$  is replaced by some power, such that the new value of  $z$  has order exactly  $2^m$ , too. Then  $b$  is replaced by  $bz$ , thus the order of the new value of  $b$  is  $2^{m-1}$  at most. Setting  $k = m$ , we get the same situation as before: the order of  $z$  is exactly  $2^k$ , and the order of  $b$  is less. Hence every time the loop is executed, the order of  $b$  is reduced, at the same time it always remains a power of 2. Hence after at most  $n$  loops, the order of  $b$  has to be 1, i.e.,  $b \equiv 1 \pmod{p}$ .

The next algorithm is our first modification of Algorithm 1.

**Algorithm 2:**

1. Set  $k = n$ ,  $z = u^q$ ,  $x = a^{(q+1)/2}$ ,  $b = a^q$ .
2. Compute  $z^2, z^{2^2}, z^{2^3}, \dots, z^{2^n}$  and store these values in an array.
3. Compute  $b^2, b^{2^2}, b^{2^3}, \dots, b^{2^n}$  and store these values in an array.
4. Set  $i = 0$ ,  $b_0 = b$ ,  $z_0 = z$

5. Let  $m$  be the least integer, such that  $b_0^{2^m} z_1^{2^m} \cdots z_i^{2^m} \equiv 1 \pmod{p}$ .
6. Set  $t = z_i^{2^{k-m-1}}$ ,  $z_{i+1} = t^2$ ,  $b = bz_{i+1}$ ,  $x = xt$ ,  $i = i + 1$ ,  $k = m$ .
7. If  $b = 1$ , stop and return  $x$ .
8. If  $i < \sqrt{n}$ , continue with (5), otherwise set  $z = z_{i+1}$  and continue with (3).

To illustrate this algorithm, we compute a solution of the congruence  $x^2 \equiv 11 \pmod{257}$ . Then we have  $q = 1$  and  $n = 8$ . As a non-residue  $\pmod{257}$ , we may choose  $u = 5$ . Hence, in the first step the variables are set to the values  $k = 8$ ,  $z = 5$ , and  $x = b = 11$ . In the second step, the powers of  $z$  are computed. We find

$z^2$	$z^4$	$z^8$	$z^{16}$	$z^{32}$	$z^{64}$	$z^{128}$	$z^{256}$
25	111	242	225	253	16	256	1

Similarly, in the third step we obtain the following table.

$b^2$	$b^4$	$b^8$	$b^{16}$	$b^{32}$	$b^{64}$	$b^{128}$	$b^{256}$
121	249	64	241	256	1	1	1

In the fourth step, we set  $i = 0$ ,  $b_0 = 11$ ,  $z_0 = 5$ .

After these initializations, we begin the loop consisting of steps (5)–(8). From the precomputed values for  $b^{2^k}$ , we see that the least  $m$  such that  $b^{2^m} \equiv 1 \pmod{257}$  is  $m = 6$ . Hence in step (6), we set  $t = z_0^2 \equiv 25$ ,  $z_1 = t^2 \equiv 111$ ,  $b = 11 \cdot 111 \equiv 193$ , and  $x = 11 \cdot 25 \equiv 18$ . These computations are somewhat facilitated by the use of the tables computed above, however, we still have to do modular multiplications. The new values  $i = 1$  and  $k = 6$  are trivial. Since  $b = 193 \not\equiv 1$ , and  $1 = i < \sqrt{n} \approx 2.8$ , we continue with step (5).

We have to find the least integer  $m$  such that  $(193)^{2^m} \equiv 1 \pmod{q}$ . The latter can be written as  $11^{2^m} 5^{2^{m+2}}$ , hence, we can use the tables above to find that  $m = 3$ , since  $253 \cdot 64 \equiv 1 \pmod{257}$ , while  $225 \cdot 249 \equiv -1 \pmod{257}$ . From this we obtain the new values  $t = 225$ ,  $z_2 = 253$  and  $x = 195$ .

Next, we have to find the least integer  $m$  such that  $(11 \cdot 111 \cdot 253)^{2^m} \equiv 1 \pmod{257}$ , that is, we have to check which of the numbers

$$11 \cdot 111 \cdot 253, \quad 121 \cdot 242 \cdot 16, \quad 249 \cdot 225 \cdot 256$$

is congruent to 1  $\pmod{257}$ . Computing these integers, we find that this is the case for  $121 \cdot 242 \cdot 16$ , hence,  $m = 1$ . From this we obtain the new values  $t = 16$ ,  $z_3 = 256$  and  $x = 36$ .

Finally, we find that

$$11 \cdot 111 \cdot 253 \cdot 256 \equiv 1 \pmod{257},$$

that is, the algorithm terminates in this step and returns  $x = 36$ , which is indeed a modular square root of 11.

In view of Algorithm 1, it is easy to see that Algorithm 2 always terminates and returns a square root, in fact, there are no essential differences between Algorithm 1 and Algorithm 2. The only major difference is that in step (5) of Algorithm 2 – which

corresponds to step (2) in Algorithm 1 – no explicit reference to  $b$  is made, but  $b$  is replaced by  $b_0 z_1 \cdots z_i$ . Of course, the numerical value of these expressions is the same, however, we claim that in the form above the algorithm needs only  $O(\log q + n^{3/2})$  modular multiplications.

Note first that for any  $i$  at any stage in the algorithm,  $z_i = u^{q \cdot 2^l}$  for some integer  $l$ , and the same is true for  $t$ . In fact, the only point where some operations are performed with these numbers is in step (6), where a certain number of squarings are performed, however, the effect of this operation is just a shift within the array of precomputed values. Hence, for any exponent  $m$  and index  $i$ ,  $z_i^{2^m}$  can be obtained by looking up in the array generated in step (2). After this remark we can compute the running time. The inner loop is performed at most  $n$  times, hence step (6) needs  $O(n)$  modular multiplications altogether. The outer loop is performed at most  $\lceil \sqrt{n} \rceil$ -times, hence step (3) requires  $n^{3/2}$  modular multiplications altogether. Step (2) requires  $n$  multiplications and is performed once, and steps (1), (4), (7) and (8) can be neglected.

Hence we have to consider step (5). The check whether for a given  $m'$  the congruence  $b^{2^{m'}} z_1^{2^{m'}} \cdots z_i^{2^{m'}} \equiv 1 \pmod{p}$  holds true, can be done using  $i$  modular multiplications, since all the powers can be obtained by looking up in the arrays generated in step (2) and (3). We already know at this stage that the congruence holds for  $m' = k$ , hence we compute the product for  $m' = k-1, k-2, \dots$ , until we find a value for  $m$  such that the product is not  $1 \pmod{p}$ . Doing so we have to check  $k-m$  values  $m'$ , hence at a given stage this needs  $(k-m)i = O((k-m)\sqrt{n})$  modular multiplications. To estimate the sum of these costs, introduce a counter  $\nu$ , which is initialized to be 0 in step (1) and raised by one in step (5), that is,  $\nu$  counts the number of times the inner loop is executed. Define a sequence  $(m_\nu)$ , where  $m_\nu$  is the value of  $m$  as found in step (5) in the  $\nu$ -th repetition of the loop. With this notation the costs of step (5) as estimated above are  $O((m_{\nu-1} - m_\nu)\sqrt{n})$ , and the sum over  $\nu$  telescopes. Since  $m_1 \leq n$ , and  $m_{\nu_1} = 1$ , where  $\nu_1$  is the value of the counter  $\nu$  when the algorithm terminates, the total cost of step (5) is  $O(n^{3/2})$ .

Putting the estimates together we see that there is a total amount of  $O(n^{3/2})$  modular multiplications. In the same way one sees that we need  $O(n^{3/2})$  look ups, and by our assumption on the costs of the latter operation we conclude that the running time of Algorithm 2 is indeed  $O(\log q + n^{3/2})$ .

Finally we describe a parallel version of Algorithm 1:

**Algorithm 3:**

1. Set  $k = n$ ,  $z = u^q$ ,  $x = a^{(q+1)/2}$ ,  $b = a^q$ .
2. Compute  $z^2, z^{2^2}, z^{2^3}, \dots, z^{2^n}$  and store these values in an array.
3. Compute  $b^2, b^{2^2}, b^{2^3}, \dots, b^{2^n}$  and store these values in an array.
4. Let  $m$  be the least integer, such that  $b^{2^m} \equiv 1 \pmod{p}$ .
5. Set  $t = z^{2^{k-m-1}}$ ,  $z = t^2$ ,  $x = xt$ ,  $k = m$ .
6. Set  $b = bz$ , compute  $b^2, b^{2^2}, \dots, b^{2^m}$  and replace the powers of  $b$  by these new values.

7. If  $b = 1$ , stop and return  $x$ , otherwise continue with step (4).

It is clear that this algorithm is equivalent to Algorithm 1, furthermore all steps with the exception of step (6) can be performed by a single processor in time  $O(\log q + n)$ . Now consider step (6). This step has to be executed at most  $n$  times, and we claim that it can be done by  $n$  processors in a single step. Indeed, since all relevant powers both of the old value of  $b$  and of  $z$  are stored, each of the powers of the new value of  $b$  can be obtained by a single multiplication, and all these multiplications can be done independently from each other on different processors. Hence, Algorithm 3 runs in time  $O(\log q + n)$  on  $n$  processors.

## References

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