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Remark on a Paper of Yu on Heilbronn's Exponential Sum

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We show that $S_h(a)=\sum_{n=1}^{p} e(an^{hp}/p^2) \ll (h, p-1)^{11/16}\, p^{7/8}$, sharpening a result of Yu.

Let p be a prime, $e(x)=e^{2\pi ix}$, and define the exponential sum $S_h(a)$ by

$$S_h(a)=\sum_{n=1}^{p} e\left(\frac{an^{hp}}{p^2}\right).$$

For a long time it was an unsolved problem whether $S_1(a)=o(p)$ uniformly in a. In 1996 Heath-Brown [1] proved that $S_1(a) \ll p^{11/12}$. Further Heath-Brown proved $S_h(a) \ll (h, p-1)^{5/4}\, p^{11/12}$ (unpublished). Yu [3] sharpened this to $S_h(a) \ll (h, p-1)\, p^{11/12}$. Recently, Heath-Brown and Konyagin [2] improved the bound for $S_1(a)$ to $S_1(a) \ll p^{7/8}$. Their results together with the method of [3] give the bound $S_h(a) \ll (h, p-1)\, p^{7/8}$. The aim of this note is to improve the dependence on h further. We will prove the following theorem.

THEOREM 1. *We have* $S_h(a) \ll (p-1, h)^{11/16}\, p^{7/8}$.

Note that we may assume $p \nmid a$, since otherwise Weil's estimate gives $S_h(a) \ll hp^{1/2}$, which together with the trivial bound $S_h(a) \leqslant p$ implies our theorem. Further we may assume $h \mid p-1$.

Our proof follows the lines of [3]; the improvement comes from the fact that we will use a nontrivial bound for the occurring sum on the characters

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of degree h. To do so, we need a little preperation. Define the polynomial f by

$$f(x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^{p-1}}{p-1}$$

and $\tilde{N}_r$ by

$$\tilde{N}_r=\#\{k\neq 0,1 \mid f(k)\equiv r \pmod p \wedge \exists y: y^h\equiv 1+1/(k-1) \pmod p\}.$$

We have $\sum_r \tilde{N}_r\leqslant \frac{p-1}{h}-1$, since the mapping $k\mapsto 1+\frac{1}{k-1}$ is bijective except for $k=1$. Further we have $\tilde{N}_r\leqslant N_r$, where N_r denotes the number of k without the restricition on $1+1/(k-1)$. Thus we can use the bounds in [2] for N_r which will be crucial to us.

To prove Theorem 1, it suffices to consider $S'_h=\sum_{n=1}^{p-1} e(an^{hp}/p^2)$. Using the orthogonality of Dirichlet characters we get the expression

$$S'_h(a)=\sum_{\chi^h=\chi_0}\sum_{y=1}^{p-1}\chi(y)\, e\left(\frac{ay^p}{p^2}\right)=:\sum_{\chi^h=\chi_0} S(a,\chi).$$

Now we compute the squared mean of the $S(a,\chi)$'s,

$$\begin{aligned}\sum_{\chi^h=\chi_0}&|S(a,\chi)|^2\\ &=h(p-1)+\sum_{\chi}\sum_{b=1}^{p-1}\sum_{k=1}^{p-1}\chi(kb)\,\bar{\chi}((k-1)\,b)\, e\left(\frac{ab^p(k^p-(k-1)^p)}{p^2}\right)\\ &=h(p-1)+\sum_{k}\left(\sum_{\chi^h=\chi_0}\chi\left(1+\frac{1}{k-1}\right)\right)S'_1(a(1-pf(k)))\\ &=h(p-1)+h\sum_{r=1}^{p}\tilde{N}_r S'_1(a(1-pr))\end{aligned}$$

since the sum over the characters equals h, if $1+\frac{1}{k-1}$ is an hth power $\pmod p$, and vanishes otherwise. The remaining sum can be estimated using the Hölder inequality,

$$\sum_{r=1}^{p-1}\tilde{N}_r S'_1(a(1-pr))\leqslant\left(\sum_{r=1}^{p}\tilde{N}_r^{4/3}\right)^{3/4}\left(\sum_{r=1}^{p}|S'_1(a(1-pr))|^4\right)^{1/4}.$$

By Theorem 2 in [2] the second sum is $\ll p^{7/2}$. To bound the first sum, we use Lemma 7 from [2] in the following form:

LEMMA 2. *Assume that there are R indices r, such that $N_r > V$. Then we have $R \ll p^2 V^{-3}$.*

Using this estimate we obtain

$$\begin{aligned}\sum_{r=1}^{p} \tilde{N}_r^{4/3} &\ll \sum_{\sqrt{ph} \leqslant 2^j \leqslant p^{2/3}} 2^{4j/3} \, \#\{r \mid N_r > 2^j\} + (ph)^{1/6} \sum_{r=1}^{p} \tilde{N}_r \\ &\ll \sum_{\sqrt{ph} \leqslant 2^j \leqslant p^{2/3}} p^2 2^{-5j/3} + p^{7/6} h^{-5/6} \\ &\ll p^{7/6} h^{-5/6}.\end{aligned}$$

Thus we obtain

$$\sum_{\chi^h = \chi_0} |S(a, \chi)|^2 \ll hp + p^{7/4} h^{3/8}.$$

Using the Cauchy–Schwarz inequality we get the estimate

$$S'_h(a) \ll hp^{1/2} + h^{11/16} p^{7/8}$$

which implies our theorem, since in the range where Theorem 1 is nontrivial, the first term is neglectable.

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