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On additive decompositions of the set of primes

By

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Abstract. We show that there is no set $\mathscr{A}$ of integers, such that $(\mathscr{P}-1) \subseteqq \mathscr{A}+\mathscr{A} \subseteqq \mathscr{P} \cup (\mathscr{P}-1)$, where $\mathscr{P}$ denotes the set of primes.

Let $\mathscr{N}$ be a set of integers. Following Wirsing[2], $\mathscr{N}$ is called additively decomposable, if there are sets $\mathscr{A}, \mathscr{B}$, such that $\mathscr{A}+\mathscr{B}=\{a+b | a \in \mathscr{A}, b \in \mathscr{B}\}=\mathscr{N}$ and both $\mathscr{A}$ and $\mathscr{B}$ have at least two elements. He showed that if $\mathscr{N}$ is probabilistically constructed with $P(n \in \mathscr{N})=\frac{1}{2}$, we have with probability 1 that $\mathscr{N}'$ is indecomposable, where $\mathscr{N}'$ is any set which equals $\mathscr{N}$ up to finitely many elements. Let $\mathscr{P}$ be the set of primes. It is still unknown, whether $\mathscr{P}$ is decomposable. For any set $\mathscr{A}$, we will use $\mathscr{A}(x)$ to denote the number of elements of $\mathscr{A}$ which are $\leqq x$. With this notation A. Hofmann and D. Wolke [1] showed that if $\mathscr{A}$ is a set such that $\mathscr{A}+(\mathscr{A}+1)=\mathscr{P}'$, then $\left(\frac{x}{\log x}\right)^{1/2} \ll \mathscr{A}(x) \ll x^{1/2}$. In this note we will show that no such $\mathscr{A}$ exists.

Theorem 1. *There is no set $\mathscr{A}$, such that $(\mathscr{P}-1)' \subseteqq \mathscr{A}+\mathscr{A} \subseteqq \mathscr{P}' \cup (\mathscr{P}'-1)$.*

Especially, if we had $\mathscr{A}+(\mathscr{A}+1)=\mathscr{P}'$, the set $\mathscr{A}$ would contradict our theorem.

Proof. Define A to be the set of residue classes (mod 30), such that $\mathscr{A}$ contains infinitely many elements from this class, B be the corresponding set for $\mathscr{A}+\mathscr{A}$ and P for $\mathscr{P}$. Then by the prime number theorem for arithmetic progressions and our assumption we get

$$\begin{aligned} P-1 &= \{0, 6, 10, 12, 16, 18, 22, 28\} \subseteqq B \\ &\subseteqq \{0, 1, 6, 7, 10, 11, 12, 13, 16, 17, 18, 19, 22, 23, 28, 29\} \\ &= P \cup (P-1). \end{aligned}$$

For every $a \in A$ we have $2a \in B$, thus

$$A \subseteqq \{0, 3, 5, 6, 8, 9, 11, 14, 15, 18, 20, 21, 23, 24, 26, 29\}.$$

Since $0 \in B$, there are elements $a_1, a_2 \in A$, such that $a_1+a_2 \equiv 0$. One easily checks that the only possibilities are $0+0$, $15+15$, $6+24$ and $9+21$. If 0 was in A, we would have $A \subseteqq B$, thus $0 \in A \subseteqq \{0, 6, 11, 18, 23, 29\}$. The only way to obtain 12 is $6+6$, and the only way to obtain 18 is $0+18$, thus $\{0, 6, 18\} \subseteqq A$. However, $6+18$ is not in B, which gives a contradiction. If 15 was in A, we would have $A+15 \subseteqq B$, thus $A \subseteqq \{3, 8, 14, 15, 21, 26\}$.

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The only way to obtain 28 is $14 + 14$, the only way to obtain 12 is $21 + 21$. However, $14 + 21 \equiv 15$ is not contained in B. If $\{6, 24\} \subseteqq A$, we get $A \subseteqq \{5, 6, 23, 24\}$, and there is no possibility to represent 6. If $\{9, 21\} \subseteqq A$, we get $A \subseteqq \{8, 9, 20, 21\}$, and again there is no possibility to represent 6.

With the same method used (mod 210) one can show that there is no $\mathscr{A}$, such that $\mathscr{P}' \subseteqq \mathscr{A} + \mathscr{A} \subseteqq \mathscr{P}' \cup (\mathscr{P}' + 1) \cup (\mathscr{P}' + 2)$, and using higher moduli one might be able to prove far more general statements. Moreover, one can replace $\mathscr{P}$ by any set $\mathscr{Q}$, such that $((\mathscr{P} \cup \mathscr{Q}) \setminus (\mathscr{P} \cap \mathscr{Q}))(x) = o\left(\sqrt{\frac{x}{\log x}}\right)$, however, this will not help deciding whether $\mathscr{P}$ is additively decomposable.

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References

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[2] E. Wirsing, Ein metrischer Satz über Mengen ganzer Zahlen. Arch. Math. **4**, 392–398 (1953).

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