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ARITHMETICAL FUNCTIONS OF THE FORM $f([g(n)])$

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Abstract. Two results on composed functions $f\big([g(n)]\big)$ are proven. First we give conditions on f and g so that the mean $\frac{1}{N}\sum_{N<n\leqq 2N} f\big([g(n)]\big)$ behaves like $\frac{1}{N}\sum_{N<n\leqq 2N} f(n)$, if $N\to\infty$, including the examples

$$\frac{1}{x}\sum_{n\leqq x}\Omega\big([n^c]\big) = \frac{1}{x}\sum_{n\leqq x}\Omega(n)+O(1),$$

$c>1$, c not an integer for $x\to\infty$. Secondly we find conditions on the real positive numbers α,β, such that $f\big([\alpha n]\big)$ and $f\big(([\alpha n],[\beta n])\big)$ are almost periodic and we compute their mean values and spectra.

1. Introduction

If $0<c<\frac{12}{11}$, Piatetski-Shapiro [7] has shown that the number of natural $n\leqq x$ for which $[n^c]$ is prime is asymptotically $\frac{x}{c\log x}$. Here $[x]$ denotes as usual the integral part of the real number x. Generally one can expect that the number-theoretical properties of the set $\big\{\,[g(n)]\,:\,n\in\mathbf{N}\big\}$ depends only on the density of this set, if $g:\,[1,\infty)\to[1,\infty)$ is a suitable function. In this direction we prove three theorems.

THEOREM 1. *Let N be a natural number, $g:\,[N,2N]\to[1,\infty)$ an $(l+2)$-times continuously differentiable function with $l\geqq 0$, and let $\alpha>1$, $c,\lambda>0$ be real constants with the properties*

$$g(x)\leqq x^c,\quad \lambda\leqq\left|g^{(l+2)}(x)\right|\leqq\alpha\lambda\quad \textit{if}\quad N\leqq x\leqq 2N.$$

Set $L=2^l$. Then we have for every additive function $f:\,\mathbf{N}\to\mathbf{C}$ with

$$\left|f(p^k)-f(p^{k-1})\right|\leqq 1\quad \textit{for all}\quad p^k$$

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and for every $0 < \delta \leqq c$

$$\sum_{N<n\leqq 2N} f\big([g(n)]\big) = \sum_{N<n\leqq 2N} f(n) + O\Big(N^{1+\delta}(\alpha^2\lambda)^{\frac{1}{4L-2}}$$

$$+ N^{1-\frac{1}{2L}+\delta}\alpha^{\frac{1}{2L}}\log N + N^{1-\frac{2}{L}+\frac{1}{L^2}+\delta}\lambda^{-\frac{1}{2L}} + cN\delta^{-1}\Big),$$

if $l \geqq 1$, *and*

$$\sum_{N<n\leqq 2N} f\big([g(n)]\big)$$

$$= \sum_{N<n\leqq 2N} f(n) + O\big(N^{1+\delta}\alpha^{2/3}\lambda^{1/3} + N^{3\delta/2}\lambda^{-1/2} + N\delta^{-1}\big),$$

if $l = 0$. *The constants implied by the O-symbol are absolute, provided that* $N^{\delta/4} \gg L\delta^{-1}$.

EXAMPLES. Let $\Omega(n)$ be the total number of prime factors of n and $c > 1$, not an integer, $l = [c] - 1$. Then in the notation of Theorem 1 we have $\alpha \ll 1$, $\lambda \asymp N^{\{c\}-1}$, hence we can choose $\delta = \frac{1-\{c\}}{8L}$ to get

$$\sum_{N<n\leqq 2N} \Omega\big([n^c]\big) = \sum_{N<n\leqq 2N} \Omega(n) + O\left(2^c\frac{N}{1-\{c\}}\right).$$

We sum over intervals of the form $(2^k, 2^{k+1}]$. Together with $\sum_{n\leqq x}\Omega(n) = x\log\log x + O(x)$ ([5], Theorem 430) we get

$$\sum_{n\leqq x}\Omega\big([n^c]\big) = x\log\log x + O\left(2^c\frac{x}{1-\{c\}}\right)$$

uniformly in c, provided that $x > 2^{2^{c+3}((c+\log^{-1}(1-\{c\}))/(1-\{c\}))}$. There is an analogous result for $\omega(n)$; but $\Omega(n^2) = 2\Omega(n)$, so that

$$\sum_{n\leqq x}\Omega(n^2) \neq \sum_{n\leqq x}\Omega(n) + O(x).$$

Note that if g is sufficiently smooth, the error term may be improved, e.g. using the theory of exponential pairs. However, for most functions g the greatest contribution to the error term stems from $N\delta^{-1}$ which cannot be improved this way.

Before we formulate the other two theorems, we need some definitions on almost periodic functions (see [8]). If $f:\mathbf{N}\to\mathbf{C}$ and $q\in[1,\infty)$, define the q-seminorm

$$\|f\|_q := \left(\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n\leqq N}\left|f(n)\right|^q\right)^{1/q}.$$

We call an arithmetical function $f:\mathbf{N}\to\mathbf{C}$ q-almost periodic ($q\in[1,\infty)$), if for each $\varepsilon>0$ there is some linear combination h over $\mathbf{C}$ of exponential functions $e_\alpha(n):=e^{2\pi i\alpha n}$, $\alpha\in\mathbf{R}$, such that $\|f-h\|_q<\varepsilon$. It is called q-limit-periodic, if one can choose exponential functions with exponents $\alpha\in\mathbf{Q}$. The space of all q-almost-periodic resp. q-limit-periodic functions is denoted by $\mathcal{A}^q$ resp. $\mathcal{D}^q$. If $f\in\mathcal{A}^q$, then the mean-value $M(f):=\lim\limits_{x\to\infty}\frac{1}{x}\sum_{n\leqq x}f(n)$ exists and so does $\hat f(\alpha):=M(fe_{-\alpha})$, $\alpha\in\mathbf{R}$. We denote $\operatorname{spec} f:=\left\{\alpha\in\mathbf{R}/\mathbf{Z}:\hat f(\alpha)\neq 0\right\}$.

Theorem 2. *Let $\alpha\in(0,\infty)$, $q\in[1,\infty)$ and $f:\mathbf{N}\to\mathbf{C}$. Define $F(n):=f\big([\alpha n]\big)$, if $n\geqq\frac{1}{\alpha}$ and 0 otherwise.*

1. *If $f\in\mathcal{A}^q$, then $F\in\mathcal{A}^q$.*
2. *If $f\in\mathcal{D}^q$ and α is irrational, then $M(F)=M(f)$ and $\operatorname{spec}F\subseteqq(\alpha\mathbf{Q})/\mathbf{Z}$.*

Remark. If f is a multiplicative function whose modulus does not exceed one, Part 2 was proven in [1].

Examples. We give two examples (see [2], p. 524).

1. Let $\alpha>0$ be irrational and $f=\mu^2$. Then F is the characteristic function of the set $\left\{n\in\mathbf{N}: n\geqq\frac{1}{\alpha},\ [\alpha n]\text{ squarefree}\right\}$ and we have $F\in\mathcal{A}^q$ $(q\geqq 1)$, $M(F)=\frac{6}{\pi^2}$, $\operatorname{spec}F\subseteqq(\alpha\mathbf{Q})/\mathbf{Z}$.
2. Let $\alpha>1$ be irrational and χ be the characteristic function of $\left\{[\alpha m]: m\in\mathbf{N},\ m\text{ squarefree}\right\}$ and $h(x)=1$ if $0<\{x\}\leqq\frac{1}{\alpha}$. Since $\chi(n)=h\big(\frac{n+1}{\alpha}\big)\mu^2\big(\big[\frac{n+1}{\alpha}\big]\big)$, we have $\chi\in\mathcal{A}^2\cdot\mathcal{A}^2\subseteqq\mathcal{A}^1$ ([8], p. 198–200) and $\operatorname{spec}\chi\subseteqq\big(\frac{1}{\alpha}\mathbf{Q}\big)/\mathbf{Z}$. Since χ is bounded, $\chi\in\mathcal{A}^q$ for every $q\geqq 1$ ([8], p. 202).

Theorem 3. *Let $f:\mathbf{N}\to\mathbf{C}$ be bounded and*

$$\delta_k := \left\| f(\cdot)-f\big((\cdot,k!)\big)\right\|_1\to 0$$

if $k\to\infty$. We define $F(n):=f\big(([\alpha n],[\beta n])\big)$ if $n\geqq\frac{1}{\alpha}$ and $n\geqq\frac{1}{\beta}$, zero otherwise ($\alpha,\beta\in(0,\infty)$).

1. *Then $F\in\mathcal{A}^q$ for $q\in[1,\infty)$.*

2. *If* $1, \alpha, \beta$ *are linearly independent over* $\mathbf{Q}$*, then* $M(F) = \sum_{n \geqq 1} \frac{f'(n)}{n^2}$*, where* $f' := f * \mu$.

3. *If* $\beta = 1$*, then*

$$M(F) = \begin{cases} \displaystyle\sum_{n \geqq 1} \frac{f'(n)}{n^2} & \alpha \textit{ irrational} \\ \displaystyle\frac{1}{b} \sum_{1 \leqq n < b} \sum_{d \mid n} \frac{f'(d)}{d} + \frac{1}{b} \lim_{k \to \infty} \sum_{d \mid k!} \frac{f'(d)}{d}, & \\ & \alpha = \dfrac{a}{b},\ b \in \mathbf{N},\ (a,b) = 1. \end{cases}$$

EXAMPLES. 1. Set $f(1) := 1$, $f(n) := 0$ if $n > 1$. Then $f' = \mu$, $\delta_k = \prod_{p \mid k!} \left(1 - \frac{1}{p}\right)$ and F (with $\beta = 1$) is the characteristic function of the set $\left\{ n \in N : \left(n, [\alpha n]\right) = 1 \right\}$. The mean-values

$$M(F) = \begin{cases} \dfrac{6}{\pi^2} & \alpha \text{ irrational} \\ \displaystyle\frac{1}{b} \sum_{1 \leqq n < b} \frac{\varphi(n)}{n}, & \alpha = \dfrac{a}{b},\ (a,b) = 1 \end{cases}$$

were computed by Watson [10], the almost-periodicity was proved in [9].

2. If $f = \mu^2$, $\beta = 1$, then F is the characteristic function of $\left\{ n \in \mathbf{N} : n \geqq \frac{1}{\alpha},\ \left(n, [\alpha n]\right) \text{ is squarefree} \right\}$, and we have

$$M(F) = \begin{cases} \dfrac{90}{\pi^4} & \alpha \text{ irrational} \\ \displaystyle\frac{1}{b} \sum_{1 \leqq n \leqq b} \sum_{t^2 \mid n} \frac{\mu(t)}{t^2} + \frac{6}{b\pi^2}, & \alpha = \dfrac{a}{b},\ b \in \mathbf{N},\ (a,b) = 1. \end{cases}$$

3. The case $\alpha = \beta = 1$ gives a criterion for almost-periodicity: every bounded function f with $\lim_{k \to \infty} \delta_k = 0$ belongs to $\mathcal{A}^q$ for all $q \geqq 1$.

2. Proof of Theorem 1

We need

LEMMA 1. *Let g, l, L, α, λ be defined as in Theorem* 1, *q be an integer. Then for $l=0$ we have*

$$\#\{n\in(N,2N]:\, q\big|\,[g(n)]\,\}=\frac{N}{q}+O\big(N\alpha^{2/3}\lambda^{1/3}q^{-1/3}+\lambda^{-1/2}q^{1/2}\big),$$

and for $l>0$ we have

$$\#\{n\in(N,2N]:\, q\big|\,[g(n)]\,\}$$
$$=\frac{N}{q}+O\left(N\left(\frac{\alpha^2\lambda}{q}\right)^{\frac{1}{4L-1}}+N^{1-\frac{1}{2L}}\alpha^{\frac{1}{2L}}\log N+N^{1-\frac{2}{L}+\frac{1}{L^2}}\left(\frac{q}{\lambda}\right)^{\frac{1}{2L}}\right).$$

PROOF. We use the notation $x=[x]+\{x\}$. Then

$$q\big|\,[g(n)]\ \Leftrightarrow\left\{\frac{g(n)}{q}\right\}<\frac{1}{q}.$$

The discrepancy D_N of the sequence $\left(\frac{g(n)}{q}\right)_{N<n\leqq 2N}$ can be estimated by the theorem of Erdős–Turán ([6], p. 114, (2.42)):

$$\left|\#\{n\in(N,2N]:\, q\big|\,[g(n)]\,\}-\frac{N}{q}\right|$$
$$=\left|\#\left\{n\in(N,2N]:\,\left\{\frac{g(n)}{q}\right\}<\frac{1}{q}\right\}-\frac{N}{q}\right|$$
$$\leqq ND_N=O\left(\min_{m\in\mathbf{N}}\left(\frac{N}{m}+\sum_{1\leqq h\leqq m}\frac{1}{h}\left|\sum_{N<n\leqq 2N}e^{2\pi ihg(n)/q}\right|\right)\right).$$

The inner exponential sum satisfies by van der Corput's theorem ([4], p. 8, Theorem 2.2) for $l=0$

$$\sum_{N<n\leqq 2N}e^{2\pi ihg(n)/q}=O\left(N\alpha\left(\frac{h\lambda}{q}\right)^{1/2}+\left(\frac{q}{h\lambda}\right)^{1/2}\right),$$

and for $l>0$ as follows ([4], p. 14, Theorem 2.8)

$$\sum_{N<n\leqq 2N} e^{2\pi i h g(n)/q}$$

$$=O\left(N\alpha^{\frac{1}{2L-1}}\left(\frac{h\lambda}{q}\right)^{\frac{1}{4L-2}}+N^{1-\frac{1}{2L}}\alpha^{\frac{1}{2L}}+N^{1-\frac{2}{L}+\frac{1}{L^2}}\left(\frac{q}{h\lambda}\right)^{\frac{1}{2L}}\right).$$

Summing over h we obtain in the first case

$$\#\left\{n\in(N,2N]:\,q\middle|\,\left[g(n)\right]\right\}$$

$$=\frac{N}{q}+O\left(\min_{m\in\mathbf{N}}\left(\frac{N}{m}+N\alpha\left(\frac{m\lambda}{q}\right)^{1/2}+\left(\frac{q}{\lambda}\right)^{1/2}\right)\right),$$

and in the second case

$$\#\left\{n\in(N,2N]:\,q\middle|\,\left[g(n)\right]\right\}=\frac{N}{q}+O\Bigg(\min_{m\in\mathbf{N}}\bigg(\frac{N}{m}+N\alpha^{\frac{1}{2L-1}}\left(\frac{m\lambda}{q}\right)^{\frac{1}{4L-2}}$$

$$+N^{1-\frac{1}{2L}}\alpha^{\frac{1}{2L}}\log m+N^{1-\frac{2}{L}+\frac{1}{L^2}}\left(\frac{q}{\lambda}\right)^{\frac{1}{2L}}\bigg)\Bigg).$$

In the first case it suffices to choose for m the nearest integer to $\left(\frac{q}{\alpha^2\lambda}\right)^{1/3}$. In the second case we certainly may assume $m\leqq N$, thus the third term is $\ll N^{1-\frac{1}{2L}}\alpha^{\frac{1}{2L}}\log N$. The first and the second term would be equal if $m=\left(\frac{q}{\alpha^2\lambda}\right)^{\frac{1}{4L-1}}$, however, m has to be an integer. We therefore choose m to be $\left[\left(\frac{q}{\alpha^2\lambda}\right)^{\frac{1}{4L-1}}\right]+1$. If $m=1$, our claim follows from the trivial bound N, otherwise changing m by an amount $\leqq 1$ does not change our estimate. Thus the error is bounded by

$$N\left(\frac{\alpha^2\lambda}{q}\right)^{\frac{1}{4L-1}}+N^{1-\frac{1}{2L}}\alpha^{\frac{1}{2L}}\log N+N^{1-\frac{2}{L}+\frac{1}{L^2}}\left(\frac{q}{\lambda}\right)^{\frac{1}{2L}}$$

which proves our claim. □

PROOF OF THEOREM 1. Since f is assumed to be additive, we have

$$\sum_{N<n\leqq 2N} f\left(\left[g(n)\right]\right)=\sum_{p^k\leqq(2N)^c} f(p^k)\#\left\{n\in(N,2N]\,\middle|\,p^k\|\left[g(n)\right]\right\}.$$

We will break up the sum on the right hand side into three sums, the first running over all $p^k \leqq N^\delta$, the second running over prime powers p^k such that $p^k > N^\delta$ and $p \geqq N^{\delta/4}$, and the last one running over prime powers p^k such that $p^k > N^\delta$ and $p < N^{\delta/4}$. We will see that the last two sums contribute only to the error term. Consider the first sum. In the sequel we will assume $l > 0$, the case $l = 0$ being similar but simpler. Using Lemma 1 we have

$$\sum_{p^k \leqq N^\delta} f(p^k)\#\{ n \in (N, 2N] \big| p^k \| [g(n)] \}$$

$$= \sum_{p^k \leqq N^\delta} \big(f(p^k) - f(p^{k-1})\big) \frac{N}{p^k}$$

$$+ O\Bigg(\sum_{p^k \leqq N^\delta} \left(N \left(\frac{\alpha^2 \lambda}{p^k} \right)^{\frac{1}{4L-1}} + N^{1-\frac{1}{2L}} \alpha^{\frac{1}{2L}} \log N + N^{1-\frac{2}{L}+\frac{1}{L^2}} \left(\frac{p^k}{\lambda} \right)^{\frac{1}{2L}} \right) \Bigg)$$

$$= \sum_{p^k \leqq N^\delta} \big(f(p^k) - f(p^{k-1})\big) \frac{N}{p^k}$$

$$+ O\Big(N^{1+\delta} (\alpha^2\lambda)^{\frac{1}{4L-1}} + N^{1-\frac{1}{2L}+\delta} \alpha^{\frac{1}{2L}} \log N + N^{1-\frac{2}{L}+\frac{1}{L^2}+2\delta} \lambda^{-\frac{1}{2L}} \Big).$$

Since the estimate of Lemma 1 is trivial for $g(n) = n$, we get

$$\sum_{p^k \leqq N^\delta} f(p^k)\#\{ n \in (N, 2N] \big| p^k \| [g(n)] \}$$

$$= \sum_{p^k \leqq N^\delta} f(p^k)\#\{ N < n \leqq 2N \big| p^k \| n \}$$

$$+ O\left(N^{1+\delta} (\alpha^2\lambda)^{\frac{1}{4L-1}} + N^{1-\frac{1}{2L}+\delta+\frac{\delta}{2L}} \alpha^{\frac{1}{2L}} \log N + N^{1-\frac{2}{L}+\frac{1}{L^2}+2\delta} \lambda^{-\frac{1}{2L}} \right).$$

Extending the sum on the right hand side to all $p^k < 2N$, we introduce an error $< \frac{2cN}{\delta}$, and the resulting sum equals $\sum_{N<n\leqq 2N} f(n)$. Thus we get

$$\sum\nolimits_1 = \sum_{N<n\leqq 2N} f(n) + O\Big(N^{1+\delta}(\alpha^2\lambda)^{\frac{1}{4L-1}} + N^{1-\frac{1}{2L}+\delta} \alpha^{\frac{1}{2L}} \log N$$

$$+ N^{1-\frac{2}{L}+\frac{1}{L^2}+2\delta} \lambda^{-\frac{1}{2L}} + \frac{cN}{\delta} \Big).$$

Finally we can replace 2δ by δ by doubling the constant implied by the O-symbol. Every $\left[g(n)\right]$ has at most $\frac{\log g(n)}{\delta/4 \log N} \leqq \frac{4c}{\delta}$ prime divisors $p > N^{\delta/4}$, counted with multiplicity, and each of them contributes at most 1 to the second sum, thus the second sum is at most $\frac{4c}{\delta}N$.

Now we consider the third sum. Let $p^k > N^\delta$ be a prime power, $p < N^{\delta/4}$. Then there is a $k' < k$ such that $N^{\delta/2} < p^{k'} \leqq N^\delta$. Obviously, the number of $n \in (N, 2N]$ such that $\left[g(n)\right]$ is divisible by p^k is at most the number of n such that $\left[g(n)\right]$ is divisible by $p^{k'}$. Using Lemma 1 again this number is

$$\ll N^{1-\delta/2} + N\left(N^{-\delta/2}\alpha^2\lambda\right)^{\frac{1}{4L-1}} + N^{1-\frac{1}{2L}}\alpha^{\frac{1}{2L}}\log N + N^{1-\frac{2}{L}+\frac{1}{L^2}}\lambda^{-\frac{1}{2L}}N^{\frac{\delta}{4L}}.$$

There are $\ll N^{\delta/4}/\log N^{\delta/4}$ primes $p \leqq N^{\delta/4}$, and if $p^k \leqq 2N$, we have $k \ll \log N$, thus there are $\ll N^{\delta/4}\log N/\log N^{\delta/4} = 4N^{\delta/4}\delta^{-1}$ such prime powers. Hence the total contribution of these terms is

$$\ll N + N^{1+\delta}(\alpha^2\lambda)^{\frac{1}{4L-1}} + N^{1-\frac{1}{2L}+\delta}\alpha^{\frac{1}{2L}}\log N + N^{1-\frac{2}{L}+\frac{1}{L^2}+\delta}\lambda^{-\frac{1}{2L}}$$

provided that $N^{\delta/4} > L\delta^{-1}$. This bound equals the error term of the first sum, thus we get for the complete sum the estimate of our theorem. □

3. Proof of Theorem 2

LEMMA 2 (oral communication by M. Peter). *Let* $h : [0,1] \to \mathbf{C}$ *have bounded variation and* $\alpha \in \mathbf{R} \setminus \mathbf{Q}$. *Then, for* $H(n) := h\big(\{\alpha n\}\big)$*:*

1. $H \in \mathcal{A}^q$ *for every* $q \geqq 1$*;*
2. $\operatorname{spec} H \subseteqq \alpha\mathbf{Z}/\mathbf{Z}$.

PROOF. Let $\sum_{k\in\mathbf{Z}} \gamma_k e_k(x)$ be the Fourier series of $h \in \mathbf{L}^q\big([0,1]\big)$ and $h_L(x) = \sum_{|k|\leqq L}\gamma_k e_k(x)$, $L \in \mathbf{N}$. The function $h_L^*(x) := h\big(\{x\}\big) - h_L(x)$ has bounded variation, so by Koksma's inequality ([6], p. 143)

$$\left|\frac{1}{N}\sum_{n\leqq N}\left|h_L^*(\alpha n)\right|^q - \int_0^1 \left|h_L^*(t)\right|^q dt\right| \leqq \operatorname{Var}\,|h_L^*|^q D_N,$$

where D_N is the discrepancy of the sequence $\big(\{\alpha n\}\big)_{n\leqq N}$. Since α is irrational, this sequence is uniformly distributed (mod 1) ([6], p. 8) and $\lim\limits_{N\to\infty} D_N$

$= 0$ ([6], p. 89). Hence

$$\left\| h_L^*(\alpha n)\right\|_q^q \leqq \overline{\lim_{N\to\infty}}\left| \frac{1}{N}\sum_{n\leqq N}\left| h_L^*(\alpha n)\right|^q - \int_0^1 \left| h_L^*(t)\right|^q dt\right| + \int_0^1 \left| h_L^*(t)\right|^q dt$$

$$= \int_0^1 \left| (h-h_L)(t)\right|^q dt.$$

The span of the exponential functions $e_k, k\in \mathbf{Z}$ is dense in $\mathbf{L}^q\big([0,1]\big)$, so

$$\left\| H - h_L(\alpha\cdot)\right\|_q = \left\| h_L^*(\alpha\cdot)\right\|_q \underset{L\to\infty}{\longrightarrow} 0.$$

It follows $H\in\mathcal{A}^q$ and $\operatorname{spec} H \subseteqq \bigcup_{L\in\mathbf{N}} \operatorname{spec} h_L(\alpha n) \subseteqq \alpha\mathbf{Z}/\mathbf{Z}$. □

Proof of Theorem 2. At first we prove that

$$e^{2\pi i r[\alpha n]}\in\mathcal{A}^q \quad \text{for every real} \quad r. \tag{1}$$

If α is rational, say $\frac{a}{b}, b\in\mathbf{N}$, then $e^{-2\pi i r\{\alpha n\}}$ has period b and so

$$e^{2\pi i r[\alpha n]} = e^{2\pi i r\alpha n}e^{-2\pi i r\{\alpha n\}}\in\mathcal{A}^q.$$

If α is irrational, we can conclude in the same way by Lemma 2.

Secondly we show:

$$\text{(2)}\quad \begin{cases}\text{For every } \varepsilon>0 \text{ we have a } \mathbf{C}\text{-linear combination } P(n)\\ \text{of exponentials } e_r,\ r\in\mathbf{R} \text{ such that } \left\| f\big([\alpha n]\big) - P\big([\alpha n]\big)\right\|_q<\varepsilon.\end{cases}$$

Since $f\in\mathcal{A}^q$, we have such a $P(n)$ with the property that $\|f-P\|_q < \varepsilon(\alpha+1)^{-1/q}$. So

$$\sum_{n\leqq N}\left| f\big([\alpha n]\big) - P\big([\alpha n]\big)\right|^q = \sum_{m\leqq \alpha N}\Bigg(\left| f(m)-P(m)\right|^q \sum_{n:\,[\alpha n]=m} 1\Bigg).$$

The inner sum is $\leqq 1+\alpha^{-1}$, so

$$\left\| f\big([\alpha\cdot]\big) - P\big([\alpha\cdot]\big)\right\|_q \leqq (\alpha+1)^{1/q}\|f-P\|_q<\varepsilon.$$

Now we can prove Part 1: If $\varepsilon > 0$ is given, we choose $P(n)$ by (2), then $P\big([\alpha n]\big) \in \mathcal{A}^q$ by (1) and so $F \in \mathcal{A}^q$.

To prove Part 2, let f be the characteristic function of some residue set $a \pmod d$, $d \in \mathbf{N}$ and $h(x) := 1$ if $\{x\} < \frac{1}{d}$, 0 otherwise. We have $f\big([\alpha n]\big) = h\big(\frac{\alpha n - a}{d}\big)$. Let $h_L(x) = \sum_{|k| \leqq L} \gamma_k e_k(x)$ with $\gamma_k := \int_0^1 h(x) e^{-2\pi i k x}\,dx$. Since α is irrational, the sequence $\big(\big\{\frac{\alpha n - a}{d}\big\}\big)_{n \in \mathbf{N}}$ is uniformly distributed and

$$\left\| f\big([\alpha n]\big) - h_L\left(\frac{\alpha n - a}{d}\right)\right\|_2^2 = \int_0^1 \big|(h - h_L)(t)\big|^2\,dt.$$

By Parseval's equality

$$\int_0^1 \big|(h - h_L)(t)\big|^2\,dt = \sum_{|k| > L} |\gamma_k|^2$$

and so

$$\lim_{L \to \infty} \left\| f\big([\alpha n]\big) - h_L\left(\frac{\alpha n - a}{d}\right)\right\|_2 = 0.$$

We get

$$M(F) = \lim_{L \to \infty} M\left(h_L\left(\frac{\alpha n - a}{d}\right)\right) = \sum_{k \in \mathbf{Z}} \gamma_k M\left(e^{2\pi i k \frac{\alpha n - a}{d}}\right) = \gamma_0,$$

$$\gamma_0 = \int_0^1 h(t)\,dt = \frac{1}{d} = M(f),$$

and $\operatorname{spec} f\big([\alpha\cdot]\big) \subseteqq \bigcup_{L \in \mathbf{N}} \operatorname{spec} h_L\big(\frac{\alpha n - a}{d}\big) \subseteqq (\alpha\mathbf{Q})/\mathbf{Z}$. So we have proved (ii) for these special characteristic functions. Since these approximate every $f \in \mathcal{D}^q$, we have the same properties in the general case, too. $\square$

4. Proof of Theorem 3

PROOF OF PART 1. Let $F_k(n) := f\big(([\alpha n],[\beta n],k!)\big)$, $k \in \mathbf{N}$. We need only show

$$F_k \in \mathcal{A}^q, \quad q \geqq 1, \tag{3}$$

$$\lim_{k\to\infty} \|F - F_k\|_q = 0. \tag{4}$$

PROOF OF (3). Let f_d be the characteristic function of $\{n \in \mathbf{N} : n \equiv 0 \pmod d\}$. Since $f = f' * 1$, we have

$$F_k(n) = \sum_{\substack{d\mid[\alpha n]\\ d\mid[\beta n]\\ d\mid k!}} f'(d) = \sum_{d\mid k!} f'(d) f_d\big([\alpha n]\big) f_d\big([\beta n]\big). \tag{5}$$

By Theorem 2, the functions $f_d\big([\alpha n]\big)$ and $f_d\big([\beta n]\big)$ belong to every $\mathcal{A}^q$, hence so does their product. So we have proved (3).

PROOF OF (4). Let $\rho_d(x) := \sum_{\substack{n\leqq x\\ ([\alpha n],[\beta n])=d}} 1$. By considerations similar to [3], p. 458 we see

$$\rho_d(x) \leqq \begin{cases} \delta\dfrac{x}{d^2} + \dfrac{\log x}{\log 2} + 1 & \text{if } x \geqq 1,\ d \in \mathbf{N},\\[2mm] 1 & \text{if } d^2 \geqq (\alpha+\beta)x, \end{cases}$$

where δ is a constant that depends only on α and β. Set $\gamma := \alpha+\beta$, then we have

$$\frac{1}{x}\sum_{n\leqq x} \big|F(n) - F_k(n)\big|^q = \frac{1}{x}\sum_{d\geqq 1} \big|f(d) - f\big((d,k!)\big)\big|^q \rho_d(x) = \sum\nolimits_1 + \sum\nolimits_2$$

with

$$\sum\nolimits_1 := \frac{1}{x}\sum_{d\leqq(\gamma x)^{1/2}} \big|f(d) - f\big((d,k!)\big)\big|^q \rho_d(x)$$

$$\leqq \delta \sum_{d\leqq(\gamma x)^{1/2}} \frac{1}{d^2}\big|f(d) - f\big((d,k!)\big)\big|^q + O(x^{-1/2}\log x)$$

and

$$\sum\nolimits_2 \leqq \frac{1}{x} \sum_{d \leqq \gamma x} \left| f(d) - f\big((d, k!)\big) \right|^q.$$

Since f is bounded, it follows

$$\|F - F_k\|_q^q = O\bigg(\sum_{d \geqq 1} \frac{1}{d^2} \left| f(d) - f\big((d, k!)\big) \right| \bigg) + O(\delta_k).$$

Since $\delta_k \to 0$, (4) is proved.

Proof of Part 2. By (4), (5) and Parseval's equality ([8], p. 207)

$$M(F) = \lim_{k \to \infty} M(F_k) = \sum_{d \geqq 1} f'(d) M\big(f_d\big([\alpha n]\big)\, f_d\big([\beta n]\big)\big)$$

$$= \sum_{d \geqq 1} f'(d) \sum_{r \in \mathbf{R}/\mathbf{Z}} M\big(f_d\big([\alpha \cdot]\big)\, e_{-r}\big)\, M\big(f_d\big([\beta \cdot]\big)\, e_r\big).$$

Since 1, α, β are linearly independent, the inner sum has, by Theorem 2, Part 2, only one nonvanishing term $r = 0$:

$$M(F) = \sum_{d \geqq 1} f'(d) M\big(f_d\big([\alpha \cdot]\big)\big)\, M\big(f_d\big([\beta \cdot]\big)\big)$$

and $M\big(f_d\big([\alpha \cdot]\big)\big) = M\big(f_d\big([\beta \cdot]\big)\big) = \frac{1}{d}$, hence $M(F) = \sum_{d \geqq 1} \frac{f'(d)}{d^2}$.

Proof of Part 3. Let $\beta = 1$. If α is irrational, then $\operatorname{spec} f_d\big([\alpha \cdot]\big) \cap \operatorname{spec} f_d\big([\beta \cdot]\big) = \{0\}$ by Theorem 2 and the mean-value formula is valid. If $\alpha = \frac{a}{b}$ with $b \in \mathbf{N}, (a, b) = 1$, then $\operatorname{spec} f_d = \left\{ \frac{l}{d} : 0 \leqq l < d \right\}$, hence

$$M(F_k) = \sum_{d | k!} f'(d) \sum_{0 \leqq l < d} \frac{1}{d} M\big(f_d\big([\alpha \cdot]\big)\, e^{-2\pi i l n/d}\big)$$

$$= \sum_{d | k!} f'(d) M\big(f_d\big([\alpha \cdot]\big)\, f_d(\cdot)\big).$$

Since the function $f_d\big([\alpha \cdot]\big)\, f_d(\cdot)$ has period db, we compute

$$M\big(f_d\big([\alpha \cdot]\big)\, f_d(\cdot)\big) = \frac{1}{db} \sum_{\substack{0 \leqq n < db \\ d | [\alpha n],\ d | n}} 1 = \frac{1}{db} \sum_{\substack{0 \leqq m < b \\ \{am/b\} < 1/d}} 1$$

$$= \frac{1}{db} \sum_{\substack{0 \leqq m < b \\ m/b < 1/d}} 1 = \begin{cases} \dfrac{1}{d^2} & \text{if } d \mid b \\ \dfrac{1}{db}\left(\left[\dfrac{b}{d}\right] + 1\right) & \text{if } d \nmid b. \end{cases}$$

So we get

$$M(F_k) = \sum_{\substack{d|k! \\ d|b}} \frac{f'(d)}{d^2} + \frac{1}{b} \sum_{\substack{d|k! \\ d\nmid b}} \frac{f'(d)}{d} \left(\left[\frac{b}{d}\right] + 1\right)$$

$$= \frac{1}{b} \sum_{d|k!} \frac{f'(d)}{d} \left[\frac{b}{d}\right] + \frac{1}{b} \sum_{\substack{d|k! \\ d\nmid b}} \frac{f'(d)}{d} = \frac{1}{b} \sum_{n<b} \sum_{\substack{d|k! \\ d|n}} \frac{f'(d)}{d} + \frac{1}{b} \sum_{d|k!} \frac{f'(d)}{d}.$$

Since $M(F) = \lim_{k \to \infty} M(F_k)$, we have proved the last formula in Theorem 3. □

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