Zero-sum sequences of medium length

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Abstract

Let $A = (a_1, \dots, a_n)$ be elements in a vector space $\mathbb{F}_p^d$. We first show that for $n \geq (d+\ell)p$ there exists a zero sum subsequence of length $\leq (d-\ell)p$, provided that $\ell \leq d/3$. We then show how the sunflower lemma can be used to prove the existence of much shorter zero sum subsequences. As an application we show e.g. that every sequence of length $2(p-1)p^{d/2}+2$ contains a zero sum subsequence of length p or $2p$. Finally we show that the existence of short zero-sums becomes much easier to prove when the exponent of the group is composite. For example we show that every sequence of length $(p+1)p^{d+1}$ in $\mathbb{Z}_{p^2}^d$ contains a zero-sum of length $\leq p^2$.

1. Introduction and results

Let $A = (a_1, \dots, a_n)$ be a sequence of elements in an abelian group G. A zero-sum subsequence is a sequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, such that $a_{i_1} + \cdots + a_{i_k} = 0$, here k will be called the length of the zero-sum. Among the basic problems in additive combinatorics is the determination of Davenport's constant $D(G)$ of a group G, that is, the least integer n, such that each sequence of length n contains a non-empty zero-sum subsequence. If G is 2-generated, or a p-group, then the precise value of $D(G)$ is known. However, in general we only have upper and lower bounds, the general believe seams to be that the upper bounds are far worse than the lower ones. Therefore obtaining improved upper bounds for $D(G)$ seems to be an interesting problem.

One method of obtaining upper bounds is by induction over subgroups. Let H be a subgroup of G, and assume we can show for some parameter k that $D(H) \leq k$, and that among every sequence of length n in G/H we can find a system of k pairwise disjoint zero-sums. Then we have $D(G) \leq n$. In fact, a sequence A of length n in G maps to a sequence $\overline{A}$ of length n in G/H. Every zero-sum in $\overline{A}$ corresponds to a subsequence of A adding up to an element in H, hence, by grouping elements in A according to the system of zero-sums in G/H we obtain a sequence of length k in H, and since $D(H) \leq k$ we can choose a zero-sum subsequence here.

Hence one is naturally lead to consider systems of disjoint zero-sums. Halter-Koch[7] introduced $D_k(G)$ as the least integer n, such that every sequence of length n over G contains a system of k disjoint zero-sums. While studying this function has given a variety of new information about $D(G)$ itself (confer e.g. [3], [1], [5]), this method suffers from the fact that the only obvious way to find large disjoint systems of zero-sums is to prove the existence of short zero-sums. Suppose one can show that in a group G every sequence of length n_0 contains a zero-sum of length ℓ. Then every sequence of length $n > n_0$ contains a system of $\lfloor n/\ell \rfloor$ disjoint zero-sums. Hence one is interested in the existence of short zero-sums.

In this article we will use two different methods to prove the existence of short zero-sums. The first method is an application of the polynomial method and generalizes [2, Lemma 6]. We will prove the following.

Theorem 1.1 *Let p be a prime, d, k be integers, such that $k \leq d/3$ and $p > d + k$. Then every sequence A of length $(k + d)p$ in $\mathbb{F}_p^d$ contains a zero-sum subsequence of length $\leq (d - k)p$.*

For the related problem of finding zero-sums of prescribed length close to dp we refer the reader to the work of Gao and Thangadurai[6] and Kubertin[8]. The obvious drawback of this theorem is the fact that it gives no information for really short zero-sums. On the other hand the required length of A is not much bigger than the trivial lower bound $D(\mathbb{F}_p^d) = d(p-1)+1$, hence this and similar results are useful rather in the endgame than in the initial stages of a proof.

Our second method is purely combinatorial and handles zero-sums of length e.g. $2p$. We postpone the statement of the somewhat technical result, and mention only the following.

Theorem 1.2 *Let p be a prime number, k, d be integers. Then every sequence A of length $k!^{2/k}(p-1)p^{d/k} + k$ in $\mathbb{F}_p^d$ contains a zero-sum subsequence Z with $|Z| \in \{p, 2p, \dots, kp\}$.*

This result is interesting only if k and p are small and d is large. In this case the usual approach is the density increment method, as introduced by Roth[10], and bounds obtained in this way are usually of the form $p^{d-\omega(d)}$, where ω is slowly tending to infinity.

This approach cannot prove any results for zero-sums of length p, however, a similar method works in the case that $\exp(G)$ is not prime. We prove the following.

Theorem 1.3 *Let G be a finite abelian group, H a subgroup such that $\exp(G) = \exp(H)\exp(G/H)$. Put $m = \min(\eta(H), |G/H|)$. Then we have*

$$\eta(G) \leq \exp(G/H)|G/H| + \exp(G)\exp(G/H)\left(|G/H|^{\exp(G/H)-1}|H|\right)^{1/\exp(G/H)}$$

Corollary 1.4 *Let p be a prime number. Then we have $\eta(\mathbb{Z}_{2p}^d) \leq 2^{d+1} + 2^{(d+1)/2}p^{(d+2)/2}$, $\eta(\mathbb{Z}_{3p}^d) \leq 3^{d+1} + 2^{2/3}3^{(d+1)/2}p^{(d+2)/2}$, $\eta(\mathbb{Z}_{p^2}^d) \leq (p+1)p^{d+1}$, and $\eta(\mathbb{Z}_2^d \oplus \mathbb{Z}_{2p}^t) \leq p^{t+1} + 2p^{t+2-t/p}2^{(d+t)/p}$.*

2. Proof of Theorem 1.2

The proof uses the polynomial method in a similar manner as in Reiher's proof of Kemnitz' conjecture (see [9]).

Theorem 2.1 (Chevalley-Warning) *Let $P_1, \dots, P_k \in \mathbb{F}_p[X_1, \dots, X_n]$ be polynomials, and assume that $\sum \deg P_i < n$. Then $|V(P_1, \dots, P_k)| \equiv 0 \pmod{p}$.*

For a sequence A, denote by $(A|k)$ the number of zero-sum sequences of length k in A.

Lemma 2.2 *Let A be a sequence of length $\geq (d+1)(p-1)+1$ in $\mathbb{F}_p^d$. Then we have*

$$(A|0) - (A|p) + (A|2p) - \cdots + (-1)^k (A|kp) \equiv 0 \pmod{p},$$

where $k = \lfloor |A|/p \rfloor$.

Proof: Put $A = (a_1, \dots, a_n)$, and write $a_i = (a_{i1}, \dots, a_{id})$ with $a_{ij} \in \mathbb{F}_p$. Then define the $d+1$ polynomials $P_1, \dots, P_{d+1} \in \mathbb{F}_p[X_1, \dots, X_n]$ as $P_j(X_1, \dots, X_n) = \sum_{i=1}^n a_{ij} X_i^{p-1}$ for $1 \leq j \leq d$, and $P_{d+1}(X_1, \dots, X_n) = \sum_{i=1}^n X_i^{p-1}$.

Suppose that $(X_1, \dots, X_n)$ is a solution of the system $P_j = 0$. The vanishing of the first d polynomials implies that $(a_i)_{X_i \neq 0}$ is a zero-sum in $\mathbb{F}_p^d$, and the last equation implies that the number of non-vanishing variables is divisible by p.

Vice versa a zero-sum sequence of length ℓ with $\ell \equiv 0 \pmod{p}$ gives rise to $(p-1)^\ell$ solutions, hence we obtain

$$(A|0) + (p-1)^p (A|p) + (p-1)^{2p} (A|2p) + \cdots = |V(P_1, \dots, P_{d+1})|,$$

and our claim now follows from Theorem 2.1. □

Lemma 2.3 *Let p be a prime number, $n, k < p$ be integers. Then we have*

$$\binom{np}{kp} \equiv \binom{n}{k} \pmod{p}.$$

Proof: We have

$$\begin{aligned}\binom{np}{kp} &\equiv \frac{(p-1)!^n}{(p-1)!^k (p-1)!^{n-k}} \cdot \frac{np \cdot (n-1)p \cdots p}{kp \cdot (k-1)p \cdots p \cdot (n-k)p(n-k-1)p \cdots p} \\ &\equiv \binom{n}{k} \pmod{p}.\end{aligned}$$

□

Lemma 2.4 *Let A be a sequence of length $\geq (d+k)p$ in $\mathbb{F}_p^d$, and suppose that $d+k < p$. Then we have for $1 \leq \ell \leq k$*

$$\binom{d+k}{d+\ell}(A|0) - \binom{d+k-1}{d+\ell-1}(A|p) + \cdots + (-1)^{d+\ell}(A|(d+\ell)p) \equiv 0 \pmod{p}.$$

Proof: Let $B \subseteq A$ be a subsequence of length ℓp. Then we have

$$(B|0) + (p-1)^p (B|p) + (p-1)^{2p} (B|2p) + \cdots + (-1)^{d+\ell}(B|(d+\ell)p) \equiv 0 \pmod{p}.$$

If we sum this congruence over all subsets of size ℓp, then a zero-sum of length mp occurs with multiplicity $\binom{(d+k-m)p}{(\ell-m)p} \equiv \binom{d+k-m}{\ell-m} \pmod{p}$, and our claim follows. □

Now let A be a sequence of length $(d+k)p$ in $\mathbb{F}_p^d$. Suppose that A contains a zero-sum subsequence Z of length $\in [d(p-1)+1, (d+k)p]$. Then this subsequence contains a zero-sum subsequence Y of strictly smaller size, hence one of Y and $Z \setminus Y$ is a zero-sum subsequence of A of length $\leq \frac{n+k}{2}$. Hence, if $k \leq n/3$, and A does not contain a zero-sum subsequence of length $\leq (n-k)p$, then we have $(A|\ell p) = 0$ for $\ell \leq n-k$ and $n+1 \leq \ell \leq n+k$. We therefore obtain from the lemma that

$$\begin{array}{cclcl}
\binom{d+k}{d+1} & + & (-1)^{d-k+1}\binom{2k-1}{k}(A|(d-k+1)p) & +\dots+ & (-1)^{d-1}\binom{k+1}{2}(A|(d-1)p) \equiv 0 \pmod p\\
\binom{d+k}{d+2} & + & (-1)^{d-k+1}\binom{2k-1}{k+1}(A|(d-k+1)p) & +\dots+ & (-1)^{d-1}\binom{k+1}{3}(A|(d-1)p) \equiv 0 \pmod p\\
\vdots & & & & \vdots \\
1 & + & (-1)^{d-k+1}(A|(d-k+1)p) & +\dots+ & (-1)^{d-1}(A|(d-1)p) \equiv 0 \pmod p
\end{array}$$

Hence we conclude that if A does not contain a zero-sum of length $\leq \min(\frac{d+k}{2}, d-k+1)$, then the system

$$\begin{pmatrix} \binom{2k-1}{k} & \binom{2k-2}{k-1} & \dots & \binom{k+1}{2} \\ \binom{2k-1}{k+1} & \binom{2k-2}{k} & \dots & \binom{k+1}{3} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \vec{x} \equiv \begin{pmatrix} \binom{d+k}{d+1} \\ \binom{d+k}{d+2} \\ \vdots \\ 1 \end{pmatrix} \pmod p \tag{1}$$

is solvable. Note that this system consists of k equations and $k-1$ variables, hence we have a real chance of proving non-solvability. We now transform the matrix by invertible column operation. The advantage of column transformation is that we do not have to keep track of the right hand side, since column operation correspond to substitutions of variables. We write $A \sim B$, if A can be transformed into B by column operations, which are invertible modulo every prime number. We have

$$\begin{pmatrix} \binom{2k-1}{k} & \binom{2k-2}{k-1} & \dots & \binom{k+1}{2} \\ \binom{2k-1}{k+1} & \binom{2k-2}{k} & \dots & \binom{k+1}{3} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \sim \begin{pmatrix} \binom{2k-2}{k} & \binom{2k-2}{k-1} & \binom{2k-3}{k-2} & \dots & \binom{k+1}{2} \\ \binom{2k-2}{k+1} & \binom{2k-2}{k} & \binom{2k-3}{k-2} & \dots & \binom{k+1}{3} \\ \vdots & \vdots & & & \vdots \\ 0 & 1 & 1 & \dots & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} \binom{2k-2}{k} & \binom{2k-3}{k-1} & \dots & \binom{k+1}{3} & \binom{k+1}{2} \\ \binom{2k-2}{k+1} & \binom{2k-3}{k} & \dots & \binom{k+1}{4} & \binom{k+1}{3} \\ \vdots & \vdots & & & \vdots \\ 1 & 1 & \dots & 1 & \binom{k+1}{k} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} \binom{2k-2}{k} & \binom{2k-3}{k-1} & \dots & \binom{k+1}{3} & \binom{k+1}{2} \\ \binom{2k-2}{k+1} & \binom{2k-3}{k} & \dots & \binom{k+1}{4} & \binom{k+1}{3} \\ \vdots & \vdots & & & \vdots \\ 1 & 1 & \dots & 1 & \binom{k+1}{k} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} \binom{k+1}{k} & \binom{k+1}{k-1} & \dots & \binom{k+1}{3} & \binom{k+1}{2} \\ 1 & \binom{k+1}{k} & \dots & \binom{k+1}{4} & \binom{k+1}{3} \\ 0 & 1 & \dots & \binom{k+1}{5} & \binom{k+1}{4} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & 1 & \binom{k+1}{k} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Hence solvability of the system (1) is equivalent to the solvability of the system

$$\begin{pmatrix} \binom{k+1}{k} & \binom{k+1}{k-1} & \dots & \binom{k+1}{3} & \binom{k+1}{2} \\ 1 & \binom{k+1}{k} & \dots & \binom{k+1}{4} & \binom{k+1}{3} \\ 0 & 1 & \dots & \binom{k+1}{5} & \binom{k+1}{4} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & 1 & \binom{k+1}{k} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \vec{y} \equiv \begin{pmatrix} \binom{d+k}{d+1} \\ \binom{d+k}{d+2} \\ \binom{d+k}{d+3} \\ \vdots \\ \binom{d+k}{d+k-1} \\ 1 \end{pmatrix} \pmod p \tag{2}$$

It is clear that solving the last $k-1$ equations gives for x_j a polynomial in d of degree $k-1-j$. However, it is not so obvious that we can give the solution of this system in closed form. We show the following.

Lemma 2.5 *Let $\vec{y}$ be the unique solution of the system (2) with the first row deleted. Then we have $y_j = \binom{d-1}{k-1-j}$.*

Proof: For $j = k-1$ we obviously have $y_{k-1} = 1$, and our claim is true. Now assume it is already proven for $j+1, j+1, \dots, k-1$. Then the last equation containing y_j becomes

$$\begin{aligned} y_j &= \binom{d+k}{d+j+1} - \binom{k+1}{k}\binom{d-1}{k-j-2} - \binom{k+1}{k-1}\binom{d-1}{k-j-3} - \dots - \binom{k+1}{j+2}\binom{d-1}{0} \\ &= \binom{d+k}{k-j-1} - \binom{k+1}{1}\binom{d-1}{k-j-2} - \binom{k+1}{2}\binom{d-1}{k-j-3} - \dots - \binom{k+1}{k-j-1}\binom{d-1}{0}. \end{aligned}$$

Now $\binom{d+k}{k-j-1}$ counts the $(k-j-1)$-element subsets of a $(d+k)$-set. If we write the $(d+k)$-set as the union of a set of $d-1$ elements and a set of $k+1$ elements, and distinguish for the number of elements of the subset contained in the $(d-1)$-set, we find that the right hand side of the last equation equals $\binom{d-1}{k-j}$, and our claim follows by induction. □

Putting these values into the first equation we obtain

$$\binom{k+1}{k}\binom{d-1}{k-2} + \binom{k+1}{k-1}\binom{d-1}{k-3} + \dots + \binom{k+1}{2}\binom{d-1}{0} \equiv \binom{d+k}{d+1} \pmod p,$$

and repeating the argument in the proof of the lemma we find that this is equivalent to $\binom{d-1}{k-1} \equiv 0 \pmod p$, which can only happen for $p < d$. Hence, if $p > d+k$ we find that the system (1) is unsolvable, and therefore the assumption about A is wrong. This contradiction proves Theorem 1.2.

3. Short zero-sums via the sunflower lemma

Suppose that $a_1, \dots, a_n$ is a sequence over $\mathbb{F}_2^d$. If $\binom{n}{2} > 2^d$, then there exist indices i_1, i_2, i_3, i_4 with $i_1 \neq i_2$, $i_3 \neq i_4$, $\{i_1, i_2\} \neq \{i_3, i_4\}$, such that $a_{i_1} + a_{i_2} = a_{i_3} + a_{i_4}$. If $i_1, \dots, i_4$ are all different, then $a_{i_1} + a_{i_2} + a_{i_3} + a_{i_4}$ is a zero-sum subsequence of length 4. If they are not all different, say $i_1 = i_3$, then $a_{i_2} = a_{i_4}$, and $a_{i_2} + a_{i_4}$ is a zero-sum of length 2. In any case

$\binom{n}{2} > 2^d$ implies the existence of a zero sum of length ≤ 4. Freeze and Schmid[5] showed that this argument can be applied with surprising success to groups containing a large $\mathbb{F}_2$-vector space as a subgroup.

Here we generalize this approach to arbitrary primes. The problem for $p > 2$ is that two equal terms in the sequence do not immediately give a zero-sum, hence, one has to consider the case of intersections between sums in more detail. We do so with the help of the sunflower lemma, introduced by Erdős and Rado[4].

Lemma 3.1 (Sunflower lemma) *Let s, ℓ be integers, and put $N = s!(\ell-1)^s+1$. Let $X_1, \dots, X_N$ be sets satisfying $|X_i| = s$. Then there exist indices $i_1, \dots, i_\ell$ and a set Y, such that $X_{i_j} \cap X_{i_k} = Y$ for all $j < k \leq \ell$.*

The bound $s!(\ell-1)^s + 1$ is not best possible, however, it is unclear what the correct magnitude of N is. Erdős and Rado conjectured that the correct magnitude should be $C^s(\ell-1)s$ for some constant C, however, in our context such an improvement would only influence the exponent of $\exp(G/H)$, which is of small order anyway. We define $S(s, \ell)$ as the smallest N for which the statement of the lemma holds true for a particular choice of s and ℓ. Using this function we can state our first main theorem. For a set I of integers and a group G we denote by $\mathfrak{S}_I(G)$ the least n, such that every sequence A of length n of elements in G contains a zero-sum of length ℓ for some integer $\ell \in I$. We write $\mathfrak{s}(G)$ in place of $\mathfrak{s}_{\exp(G)\}}(G)$, $\eta(G)$ in place of $\mathfrak{s}_{\{1,\dots,\exp(G)\}}(G)$, and $\eta_\ell(G)$ in place of $\mathfrak{s}_{\{1,\dots,\ell\}}(G)$

Theorem 3.2 *Let p be a prime number, k, d, d', N be integers, and assume that $\binom{N}{k} \geq p^{d-d'} S(k, \mathfrak{s}(\mathbb{Z}_p^{d'}))$. Then we have $\mathfrak{s}_{\{p,2p,\dots,kp\}} \leq N$.*

Proof: Let A be a sequence of length N. Fix a subgroup $U \cong \mathbb{Z}_p^{d'}$, denote by $\pi : \mathbb{Z}_p^d \to \mathbb{Z}_p^{d-d'}$ the projection mapping U to 0, and put $\overline{A} = \pi(A)$. Consider all k-subsets $X_1, \dots, X_{\binom{N}{k}}$ of $\overline{A}$. From the definition of $S(k, e)$ and the assumption on the size of $\binom{N}{k}$ we find that there are indices $i_1, \dots, i_{\mathfrak{s}(\mathbb{Z}_p^{d'})}$ and a set Y such that $\sum_{x \in X_{i_j}} x$ is independent of j, and $X_{i_j} \cap X_{i_k} = Y$. Put $Z_j = X_{i_j} \setminus Y$. Then $\sum_{z \in Z_i} z$ is independent of j, and all Z_j have the same length $\ell \leq k$. Taking pre-images under π we obtain a sequence of length $\mathfrak{s}(\mathbb{Z}_p^{d'})$ contained in one coset of U, within this sequence we can choose a zero-sum of length p. Hence there are indicies $j_1, \dots, j_p$, such that $\sum_{z \in Z_1 \cup \dots \cup Z_p} z = 0$, and $Z_1 \cup \dots \cup Z_p$ is a zero sum subsequence of A of length ℓp. □

Corollary 3.3 *Let p be a prime number, and k, d be integers. Then we have*

$$\begin{aligned}
\mathfrak{s}_{\{p,2p,\dots,kp\}}(\mathbb{Z}_p^d) &\leq k!^{2/k}(p-1)p^{d/k} + k, \\
\mathfrak{s}_{\{p,2p,\dots,kp\}}(\mathbb{Z}_p^d) &\leq k!^{2/k}(2p-2)p^{(d-1)/k} + k \\
\mathfrak{s}_{\{p,2p,\dots,kp\}}(\mathbb{Z}_p^d) &\leq k!^{2/k}(4p-4)p^{(d-2)/k} + k.
\end{aligned}$$

If $\mathfrak{s}(\mathbb{Z}_p^3) = 9p - 8$, then

$$\mathfrak{s}_{\{p,2p,\dots,kp\}}(\mathbb{Z}_p^d) \leq k!^{2/k}(9p-9)p^{(d-3)/k} + k.$$

In particular the last inequality holds true for $p = 3$ and $p = 5$.

Proof: We have $\binom{N}{k} > \frac{(N-k+1)^k}{k!}$, hence $\binom{N}{k} \geq p^{d-d'} S(k, \mathfrak{s}(\mathbb{Z}_p^{d'}))$ holds true, provided that $(N-k+1)^k \geq k!^2 p^{d-d'}(\mathfrak{s}(\mathbb{Z}_p^{d'})-1)^k+1$. Our claim now follows from the results $\mathfrak{s}_{\{p\}}(1)=p$, $\mathfrak{s}(\mathbb{Z}_p)=2p-1$, and $\mathfrak{s}(\mathbb{Z}_p^2)=4p-3$. □

Clearly our method can only handle zero-sums of length $2p$ or longer, however, this is not a deficit of the method but a real phenomenon. In fact, we have $\eta_{2p-1}(\mathbb{F}_p^d) \geq \mathfrak{s}(\mathbb{F}_p^{d-1})$. To see this take a sequence A over $\mathbb{F}_p^{d-1}$ without zero sums of length p, and append a digit 1 to each element of A to obtain a sequence A' over $\mathbb{F}_p^d$ with $|A|=|A'|$. Considering the last coordinate we find that every zero-sum of A' has length divisible by p, while considering the first $d-1$ coordinates we find that A' has no zero-sum of length equal to p. Hence A' has no zero-sums of length $\leq 2p-1$, and since this construction works for all A, we obtain $\eta_{2p-1}(\mathbb{F}_p^d) \geq \mathfrak{s}(\mathbb{F}_p^{d-1})$. Since it may well be that $\mathfrak{s}(\mathbb{F}_p^d)$ is not much smaller than p^d, we see that we cannot expect a good bound for zero sums of length $2p-1$ or less.

We now turn to the proof of Theorem 1.3. Let A be $\pi: G \to G/H$ be the canonical projection. For $g \in G/H$ let A_g be the subsequence consisting of elements A of A with $\pi(a)=g$, and put $n_g=|A_g|$. If we pick $\exp(G/H)$ elements of A which map to the same element under π, their sum is an element of H. In this way we obtain $\sum_{g\in G/H}\binom{n_g}{\exp(G/H)}$ elements in H. If this quantity supersedes $(S(\exp(G/H), \exp(H))-1)|H|$, then there exists some $h \in H$, such that there are $S(\exp(G/H), \exp(H))$ different $\exp(G/H)$-tuple with sum h. Among these tuples we find a sunflower with $\exp(H)$ petals, which gives a zero-sum of length dividing $\exp(G/H)\exp(H)=\exp(G)$. The function $x \mapsto \binom{x}{k}$ is convex for every integral $k \geq 1$, hence

$$\sum_{g\in G/H}\binom{n_g}{\exp(G/H)} \geq \binom{|A|/|G/H|}{\exp(G/H)}|G/H| \geq \frac{(|A|-\exp(G/H)|G/H|)^{\exp(G/H)}}{\exp(G/H)!|G/H|^{\exp(G/H)-1}}.$$

Hence A contains a zero-sum of length dividing $\exp(G)$, provided that

$$(|A|-\exp(G/H)|G/H|)^{\exp(G/H)} \geq \exp(G/H)!|G/H|^{\exp(G/H)-1}S(\exp(G/H), \exp(H))|H|.$$

From this we obtain

$$\begin{aligned}\eta(G) &\leq \exp(G/H)|G/H| + \left(\exp(G/H)!^2\exp(H)^{\exp(G/H)}|G/H|^{\exp(G/H)-1}|H|\right)^{1/\exp(G/H)}\\ &\leq \exp(G/H)|G/H| + \exp(G)\exp(G/H)\left(|G/H|^{\exp(G/H)-1}|H|\right)^{1/\exp(G/H)}.\end{aligned}$$

Since we are most interested in the case of groups of large rank, the dependence on the exponent of the groups does not really matter. Ignoring these terms for the moment we find that the most important contribution is $|G/H|$, while $|H$ plays a minor rôle. This is surprising, since from an algebraic point of view $|H|$ and $|G/H|$ are symmetric.

The corollary follows from Theorem 1.3, in the case of $\mathbb{Z}_{2p}$ and $\mathbb{Z}_{3p}$ we took $H=\mathbb{Z}_p^d$, and used the better formula in which $n!$ is not replaces by n^n.

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