New York Journal of Mathematics

New York J. Math. **11** (2005) 205–224.

Divisibility properties of subgroup numbers for the modular group

T. W. Müller and **J.-C. Schlage-Puchta**

Abstract. Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ be the classical modular group. It has been shown by Stothers (Proc. Royal Soc. Edinburgh **78A**, 105–112) that s_n, the number of index n subgroups in Γ, is odd if and only if $n+3$ or $n+6$ is a 2-power. Moreover, Stothers (loc. cit.) also showed that f_λ, the number of free subgroups of index 6λ in Γ, is odd if and only if $\lambda+1$ is a 2-power. Here, these divisibility results for f_λ and s_n are generalized to congruences modulo higher powers of 2. We also determine the behaviour modulo 3 of f_λ. Our results are naturally expressed in terms of the binary respectively ternary expansion of the index.

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1. Introduction and results

Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ be the classical modular group. We denote by s_n the number of index n subgroups in Γ, and by f_λ the number of free subgroups in Γ of index 6λ. These days, quite a lot is known concerning the subgroup arithmetic of Γ. Newman [5, Theorem 4] gave an asymptotic formula for s_n; for a more general and more precise result see [3, Theorem 1]. Based on numerical computations of Newman, Johnson conjectured that s_n is odd if and only if $n = 2^a - 3, a \geq 2$ or $n = 2^a - 6, a \geq 3$. This conjecture was first proved by Stothers [6]. He first used coset diagrams to establish a relation between s_n and f_λ for various λ in the range $1 \leq \lambda \leq \frac{n+4}{6}$, and then showed that f_λ is odd if and only if $\lambda = 2^a - 1, a \geq 1$. The parity pattern for f_λ found by Stothers has been shown to hold for a larger class of

Received August 26, 2004.

Mathematics Subject Classification. 20E.

Key words and phrases. modular group, subgroup numbers, congruences.

ISSN 1076-9803/05

virtually free groups, including free products $\Gamma = G_1 *_S G_2$ of two finite groups G_i with an amalgamated subgroup S of odd order, whose indices $(G_i : S)$ satisfy

$$\{(G_1 : S), (G_2 : S)\} = \{2, 3\} \text{ or } \{2, 4\};$$

cf. [2, Prop. 6]. An alternative proof of Johnson's conjecture making use of a new recurrence relation for s_n was given by Godsil, Imrich, and Razen [1]. The principal purpose of the present paper is to generalize the divisibility results for f_λ and s_n mentioned to congruences modulo higher powers of 2. We also describe the behaviour of f_λ modulo 3. For a prime p and a positive integer n denote by $\mathfrak{s}_p(n)$ the sum of digits in the expansion of n to base p. Our main results are as follows.

Theorem 1. (i) *If $\lambda \geq 6$ is even, then $64 | f_\lambda$.*

In (ii)–(vi) *below, let $\lambda > 20$ be an odd integer.*

(ii) *If $\mathfrak{s}_2(\lambda + 1) = 1$, that is, $\lambda = 2^a - 1$ for some a, then $f_\lambda \equiv 13\,(16)$.*

(iii) *If $\mathfrak{s}_2(\lambda + 1) = 2$, write $\lambda = 2^a + 2^b - 1, a > b \geq 1$. Then*

$$f_\lambda \equiv \begin{cases} 14, & b = 1 \\ 6, & b = 2 \\ 2, & a = b + 1 \\ 6, & a = b + 2 \\ 14, & \text{otherwise.} \end{cases} \pmod{16}$$

(iv) *If $\mathfrak{s}_2(\lambda + 1) = 3$, write $\lambda = 2^a + 2^b + 2^c - 1$, where $a > b > c \geq 1$. Assume that precisely k of the equations $a = b + 1$, and $b = c + 1$ hold, $k = 0, 1, 2$. Then*

$$f_\lambda \equiv \begin{cases} 4 \pmod{16}, & k \equiv 0\,(2) \\ 12 \pmod{16}, & k \equiv 1\,(2). \end{cases}$$

(v) *If $\mathfrak{s}_2(\lambda + 1) = 4$, then $f_\lambda \equiv 8\,(16)$.*

(vi) *If $\mathfrak{s}_2(\lambda + 1) \geq 5$, then $f_\lambda \equiv 0\,(16)$.*

The regular behaviour of the function f_λ described in Theorem 1 breaks down for $\lambda < 20$. Here the values modulo 16 are as follows:

λ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
f_λ	5	12	1	0	2	0	5	0	6	0	2	0	4	0	5	0	6	0	6.

Theorem 2. *Let $n \geq 22$ be an integer. Then we have modulo 8:*

$$s_n \equiv \begin{cases} 1, & n = 2^a - 3 \\ 5, & n = 2^a - 6 \\ 2, & n = 3 \cdot 2^a - 3, 3 \cdot 2^a - 6 \\ 6, & n = 2^a + 2^b - 3, 2^a + 2^b - 6, 2^a + 3,\ a \geq b + 2, b \geq 2 \\ 4, & n = 2^a + 2^b + 2^c - 6,\ a > b > c \geq 2, \\ & n = 2^a + 2^b + 2^c - 3,\ a > b > c \geq 2, b \geq 4, \\ & n = 2^a + 2^b + 3,\ a > b \geq 2 \\ 0, & \text{otherwise.} \end{cases}$$

Again, for smaller values of n the behaviour of s_n mod 8 is irregular:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
s_n	1	1	4	0	5	6	2	0	0	1	6	4	1	6	0	0	2	6	6	0	6	4.

Based on a certain amount of numerical computation, we believe that the discrepancy between odd and even values of λ visible in Theorem 1 grows for larger values of λ. More precisely, we propose the following:

Conjecture 1. *For λ even we have $\nu_2(f_\lambda) \geq \lambda$ with equality occurring infinitely often, whereas for λ odd we have $\nu_2(f_\lambda) = \mathfrak{s}_2(\lambda+1) - 1$.*

Computing f_λ for $\lambda \leq 1000$, we have established Conjecture 1 in this range. The lower bound $\nu_2(f_\lambda) \geq \lambda$ for λ even appears to be close to optimal; in fact, the differences $\nu_2(f_\lambda) - \lambda$ with $\lambda \leq 1000$ and even are distributed as shown in the following table:

$\nu_2(f_\lambda) - \lambda$	0	2	3	4	5	6	7	8	9	10	11	12	13	14	16
$\#\lambda$	88	99	56	73	57	46	40	17	7	5	5	2	3	1	1.

The average for $\nu_2(s_n) - n$ is 3.792; the most prominent feature, however, is the absence of integers n with $\nu_2(s_n) - n = 1$.

We record a partial result towards Conjecture 1.

Proposition 1. (i) *For λ odd, we have $\nu_2(f_\lambda) = \mathfrak{s}_2(\lambda+1) - 1$, while for λ even we have $\nu_2(f_\lambda) \geq \mathfrak{s}_2(\lambda+1)$.*

(ii) *We have $\nu_2(s_n) \geq \frac{1}{2}\mathfrak{s}_2(n+6) + \nu_2(n) - \frac{5}{2}$.*

Moreover, the proof of Theorem 1 suggests the following:

Conjecture 2. *For every l and k there is some d and some a, such that $f_\lambda \equiv a\,(2^k)$ for all λ satisfying $\mathfrak{s}_2(\lambda+1) = l$, such that between any two consecutive 1's in the dyadic expansion of λ there are at least d zeros.*

Finally, we also give a description of the function f_λ modulo 3. By a Cantor number we shall mean a nonnegative integer whose 3-adic expansion does not contain any 1's.

Theorem 3. (i) *$f_\lambda \not\equiv 0\,(3)$ if and only if $\lambda - 1$ is a Cantor number or λ is of the form $3^{a+1}n + 2\cdot 3^a - 1$, where n is a Cantor number and $a \geq 1$. Moreover, for such λ, we have $f_\lambda \equiv -1\,(3)$ in the first case, and $f_\lambda \equiv (-1)^{a-1}\,(3)$ in the second case.*

(ii) *Denote by $N(x)$ the number of integers $\lambda \leq x$ satisfying $f_\lambda \not\equiv 0\,(3)$. Then $N(x) \asymp x^{\log 2/\log 3}$.*

Here, $f(t) \asymp g(t)$ means that $f(t)$ and $g(t)$ are of the same order of magnitude; that is, there exist positive constants c_1, c_2, such that $c_1 f(t) \leq g(t) \leq c_2 f(t)$. The function $N(x)$ displays a fractal behaviour typical of digital problems, hence, cannot be expected to satisfy a simple asymptotic law.

2. Proof of Theorem 1

Lemma 1. *The numbers f_λ satisfy the recurrence relation*

$$f_\lambda = 6\lambda f_{\lambda-1} + \sum_{\substack{\mu,\nu\geq 1\\ \mu+\nu=\lambda-1}} f_\mu f_\nu, \quad \lambda \geq 2, \tag{1}$$

with initial value $f_1 = 5$.

Proof. This is essentially Proposition 1.9 in [6] (apart from some misprints); see also [2, formula (24)]. □

Lemma 2. *For integers $a, b \geq 1$, we have $\mathfrak{s}_2(a+b) \leq \mathfrak{s}_2(a) + \mathfrak{s}_2(b)$.*

Proof. In fact, $\mathfrak{s}_2(a) + \mathfrak{s}_2(b) - \mathfrak{s}_2(a+b)$ is the number of carries in the addition $a+b$ to base 2, which is certainly nonnegative. □

The proof of Theorem 1 proceeds recursively for increasing powers of 2; we treat every step as a separate lemma.

Lemma 3. *Modulo 4, we have the congruence*

$$f_\lambda \equiv \begin{cases} 1; & \lambda = 2^a - 1,\ a \geq 1\\ 2; & \lambda = 2^a + 2^b - 1,\ a > b \geq 1\\ 0; & \text{otherwise.}\end{cases}$$

Proof. For $\lambda \leq 8$, the claimed congruence for f_λ is easily checked numerically. Let $\lambda_0 \geq 10$ be an even integer and assume that f_λ is divisible by 4 for all even $\lambda < \lambda_0$. From (1), we obtain the relation

$$f_{\lambda_0} \equiv \sum_{\substack{\mu,\nu\geq 1\\ \mu+\nu=\lambda_0-1}} f_\mu f_\nu \pmod 4.$$

Since $\lambda_0 - 1$ is odd, precisely one factor in each summand has even index. Hence, by our inductive hypothesis, each summand is divisible by 4, and our claim is proven for even values of λ. Now let $\lambda_0 \geq 9$ be odd, and assume that our claim is already proven for all integers $\lambda < \lambda_0$. From (1) and the result for even values of λ already established we deduce the relation

$$f_{\lambda_0} \equiv \sum_{\substack{\mu,\nu\geq 1\\ \mu+\nu=\lambda_0-1}} f_\mu f_\nu \equiv \sum_{\substack{\mu,\nu\geq 1\\ \mu+\nu=\lambda_0-1\\ \mu,\nu\equiv 1\ (2)}} f_\mu f_\nu \equiv f^2_{(\lambda_0-1)/2} + 2 \sum_{\substack{1\leq\mu<\nu\\ \mu+\nu=\lambda_0-1\\ \mu,\nu\equiv 1\ (2)}} f_\mu f_\nu \pmod 4.$$

By our inductive hypothesis, we may restrict the summation in the last expression further, and suppose that $\mu = 2^a - 1, \nu = 2^b - 1$. If $\mathfrak{s}_2(\lambda+1) \geq 3$ or $\mathfrak{s}_2(\lambda+1) = 1$, the equation $\mu + \nu = \lambda - 1$ is not solvable in such integers, and the sum displayed last vanishes modulo 4. If $\mathfrak{s}_2(\lambda+1) = 2$, then $\lambda = 2^a + 2^b - 1$, say, and there is a unique solution of the equation $2^x + 2^y = 2^a + 2^b$ in integers $x > y \geq 1$; hence, in this case the sum is odd. From this we derive the congruence

$$f_{\lambda_0} \equiv f^2_{(\lambda_0-1)/2} + \begin{cases} 2, & \mathfrak{s}_2(\lambda+1) = 2\\ 0, & \text{otherwise,}\end{cases} \pmod 4$$

which implies our claim. □

Lemma 4. *For an odd integer λ, we have the congruence*

$$f_\lambda \equiv \begin{cases} 5, & \mathfrak{s}_2(\lambda+1)=1\\ 6, & \mathfrak{s}_2(\lambda+1)=2,\ \lambda\neq 3\cdot 2^a-1\\ 2, & \mathfrak{s}_2(\lambda+1)=2,\ \lambda=3\cdot 2^a-1\\ 4, & \mathfrak{s}_2(\lambda+1)=3\\ 0, & \mathfrak{s}_2(\lambda+1)\geq 4,\end{cases} \pmod 8$$

with the exception $f_3\equiv 1\,(8)$.

Proof. For $\lambda\leq 19$ the statement can be read of the table following Theorem 1, so let $\lambda\geq 20$ be an odd integer. As in the proof of Lemma 3, we find that

$$f_\lambda \equiv f_{(\lambda-1)/2}^2 + 2\sum_{\substack{1\leq\mu<\nu\\ \mu+\nu=\lambda-1\\ \mu,\nu\equiv 1\ (2)}} f_\mu f_\nu \pmod 8.$$

In view of Lemma 3, all summands satisfying $\mathfrak{s}_2(\mu+1)+\mathfrak{s}_2(\nu+1)\geq 4$ do not contribute to the right-hand side. Consider first the case where $\lambda=2^a-1$. If $\mathfrak{s}_2(\mu+1)=\mathfrak{s}_2(\nu+1)=1$, then $\mu+1$ and $\nu+1$ are powers of 2 adding up to 2^a; hence, they have to be equal, contradicting the summation condition. If one of $\mu+1$, $\nu+1$ has sum of digits equal to 2, and the other has sum of digits equal to 1, these values represent a solution of the equation $2^x+2^y+2^z=2^a$. Up to permutation of the variables, the only solution of this equation is $x=y=z-1=a-2$, and the only possibility arising from this solution is $\mu=2^{a-2}-1$, $\nu=2^{a-1}+2^{a-2}-1$. Note that $a\geq 3$, hence we can apply Lemma 3 to obtain

$$f_\lambda\equiv f_{(\lambda-1)/2}^2+4\equiv 5 \pmod 8.$$

Next consider the case where $\mathfrak{s}_2(\lambda+1)=2$, that is, $\lambda=2^a+2^b-1$ with $a>b\geq 1$. Only pairs of indices (μ,ν) satisfying $\mathfrak{s}_2(\mu)+\mathfrak{s}_2(\nu)\leq 3$ contribute to the sum; consider first the possibility that $\mathfrak{s}_2(\mu)+\mathfrak{s}_2(\nu)=3$. We have to consider solutions of the equation $2^x+2^y+2^z=2^a+2^b$. Up to permutation of variables, all solution are of the form $(a,b-1,b-1)$ or $(a-1,a-1,b)$. However, since μ and ν are odd integers, and $2^x,2^y,2^z$ are the terms of their dyadic expansion, the first solution can only occur if $b\geq 2$, and the second solution can only occur if $a\geq b+2$. By Lemma 3, we find that the contribution of such terms is 4 modulo 8, if $\lambda=3\cdot 2^a-1$ or $\lambda=2^a+1$, and 0 modulo 8 otherwise. If $b\neq 1$, then $(\lambda-1)/2$ is an odd integer with $\mathfrak{s}_2((\lambda-1)/2)=2$; hence, $f_{(\lambda-1)/2}^2\equiv 4\,(8)$. On the other hand, for $b=1$, $(\lambda-1)/2$ is a power of two, and therefore $f_{(\lambda-1)/2}^2\equiv 0\,(8)$. Finally consider solutions $\mu+\nu=\lambda-1$ satisfying $\mathfrak{s}_2(\mu+1)=\mathfrak{s}_2(\nu+1)=1$. Then μ and ν are given by the dyadic digits of λ, and, using Lemma 3 again, we obtain that the contribution of these solutions is 2 modulo 8. Adding up the various contributions, we obtain $f_\lambda\equiv 2\,(8)$ if $a=b+1$, and $f_\lambda\equiv 6\,(8)$ otherwise.

Now assume that $\mathfrak{s}_2(\lambda+1)=3$. We can restrict the sum to pairs (ν,μ) satisfying $\mathfrak{s}_2(\mu+1)+\mathfrak{s}_2(\nu+1)\leq 3$; by Lemma 2, the case $\mathfrak{s}_2(\mu+1)+\mathfrak{s}_2(\nu+1)=2$ cannot occur either. Write $\lambda=2^a+2^b+2^c-1$ with $a>b>c\geq 1$. Then the only solutions of $\mu+\nu=\lambda-1$ satisfying $\mu<\nu$ and $\mathfrak{s}_2(\mu+1)+\mathfrak{s}_2(\nu+1)=3$ are $(2^c,2^a+2^b)$, $(2^b,2^a+2^c)$ and $(2^b+2^c,2^a)$. In each case, the corresponding summand is 2 modulo 4; hence, the sum is 2 modulo 4. Now $(\lambda-1)/2$ is either even or satisfies

$\mathfrak{s}_2((\lambda-1)/2+1)=3$, depending on whether $c=1$ or not. In each case, $f_{(\lambda-1)/2}$ is divisible by 4, and we obtain $f_\lambda \equiv 4\,(8)$.

Finally, consider the case $\mathfrak{s}_2(\lambda+1)\geq 4$. Then $f_{(\lambda-1)/2}$ is divisible by 4, and every solution of the equation $\mu+\nu=\lambda-1$ satisfies $\mathfrak{s}_2(\mu+1)+\mathfrak{s}_2(\nu+1)\geq 4$. Hence, $f_\lambda\equiv 0\,(8)$, and the lemma is proven in all cases. □

Proof of Theorem 1. We begin by first establishing (i) modulo 16; we then proceed to the proofs of (ii)–(vi), making use of the available information for even arguments, before coming back to (i) and finishing the proof modulo 64, using all the information concerning values of f_λ previously established.

(i), preliminary discussion: we argue by induction on λ. For $\lambda\leq 20$, the claim can be checked numerically. Let $\lambda>20$ be even, and assume that f_ν is divisible by 64 for ν even and in the range $6\leq\nu\leq\lambda-2$. In the sum on the right-hand side of (1), each summand consists of one factor with odd index, and one factor with even index. Hence, we obtain the congruence

$$(2)\quad f_\lambda \equiv 6\lambda f_{\lambda-1}+2f_2f_{\lambda-3}+2f_4f_{\lambda-5}\equiv 6\lambda f_{\lambda-1}+56f_{\lambda-3}+32f_{\lambda-5}\pmod{64}.$$

If $\mathfrak{s}_2(\lambda)\geq 3$, then $\mathfrak{s}_2(\lambda-2)\geq 2$, and hence, by Lemma 3, $f_\lambda\equiv 0\ (16)$. If, on the other hand, $\lambda=2^a-2$, then $f_{\lambda-3}\equiv 1\ (2)$ and $f_{\lambda-1}\equiv 2\ (4)$, thus again $f_\lambda\equiv 0\ (16)$. Now suppose that $\mathfrak{s}_2(\lambda)=2$ and that $\lambda-2$ is not a 2-power. Then $\lambda\equiv 0\ (4)$, both $f_{\lambda-1}$ and $f_{\lambda-3}$ are even, and hence again $f_\lambda\equiv 0\ (16)$ by (2). Thus, in every case, we have $f_\lambda\equiv 0\ (16)$. We postpone the discussion of f_λ modulo 64 until we have gathered enough information concerning values of f for odd arguments.

For the proof of (ii)–(vi), note that Equation (1) reduces to

$$(3)\qquad f_\lambda\equiv (f_{\frac{\lambda-1}{2}})^2+2\sum_{\substack{\mu,\nu\equiv 1\,(2)\\ \mu<\nu\\ \mu+\nu=\lambda-1}} f_\mu f_\nu,\qquad \pmod{16}.$$

If $\mathfrak{s}_2(\nu+1)+\mathfrak{s}_2(\mu+1)\geq 5$, then $f_\mu f_\nu$ is divisible by 8; thus, we only have to consider solutions of the equation $\mu+\nu=\lambda-1$ satisfying $\mathfrak{s}_2(\nu+1)+\mathfrak{s}_2(\mu+1)\leq 4$. We now discuss each of the assertions (ii)–(vi) separately.

(ii) Write $\lambda=2^a-1$. Consider the set of all pairs of indices (μ,ν) in the right-hand sum in (3) satisfying $\mathfrak{s}_2(\nu+1)+\mathfrak{s}_2(\mu+1)\leq 4$. If $\mathfrak{s}_2(\nu+1)+\mathfrak{s}_2(\mu+1)=4$, we obtain the equation

$$2^x+2^y+2^z+2^u=2^a,$$

which has the solutions $(a-2,a-2,a-2,a-2)$ and, up to permutation of the variables, $(a-1,a-2,a-3,a-3)$. The former solution cannot be realized by terms of the dyadic expansion of two integers, whereas the latter one leads to the possibilities $\mu=2^{a-3}-1$, $\nu=2^{a-1}+2^{a-2}+2^{a-3}-1$ and $\mu=2^{a-2}+2^{a-3}-1$, $\nu=2^{a-1}+2^{a-3}-1$. Since $a\geq 4$, we find in both cases $f_\mu f_\nu\equiv 4\,(8)$; hence, the contribution of pairs (μ,ν) with $\mathfrak{s}_2(\mu{+}1){+}\mathfrak{s}_2(\nu{+}1)$ is 0 modulo 16. Pairs (μ,ν) with $\mathfrak{s}_2(\nu+1)+\mathfrak{s}_2(\mu+1)=3$ correspond to solutions of the equation $2^x+2^y+2^z=2^a$. Up to permutation of variables, the only solution is $(a-1,a-2,a-2)$, leading to $\mu=2^{a-2}-1$, $\nu=2^{a-1}+2^{a-2}-1$. From Lemma 4, we deduce that $f_\mu f_\nu\equiv 2\,(8)$. Finally, pairs satisfying $\mathfrak{s}_2(\nu+1)+\mathfrak{s}_2(\mu+1)=2$ correspond to solutions of $2^x+2^y=2^a$. The only solution of this equation is $(a-1,a-1)$, which yields $\mu=\nu$, contradicting the summation condition. We find that the sum on the right-hand side of (3) is 2 modulo 8. Note that $(\lambda-1)/2$ is always odd, hence we can use Lemma 4 to obtain $f_{(\lambda-1)/2} \equiv 5\,(8)$, implying $(f_{(\lambda-1)/2})^2 \equiv 9\,(16)$. From these computations our claim follows.

(iii) Write $\lambda = 2^a + 2^b - 1$ with $a > b \geq 1$. If $b \geq 2$, then $(\lambda-1)/2$ is odd and $\mathfrak{s}_2((\lambda-1)/2) = 2$; hence $f_{(\lambda-1)/2} \equiv 2\,(8)$ if $a = b+1$, and $f_{(\lambda-1)/2} \equiv 6\,(8)$ if $a \geq b+2$. If $b = 1$, then $(\lambda-1)/2$ is even, and $f_{(\lambda-1)/2} \equiv 0\,(8)$. Hence, we obtain

$$(f_{(\lambda-1)/2})^2 \equiv \begin{cases} 4, & b \geq 2 \\ 0, & b = 1. \end{cases} \pmod{16}$$

Next, consider terms satisfying $\mathfrak{s}_2(\nu+1)+\mathfrak{s}_2(\mu+1) = 4$ in the sum on the right-hand side of (3). These terms correspond to solutions of the equation $2^x + 2^y + 2^z + 2^u = 2^a + 2^b$. This equation has the solutions $(a, b-1, b-2, b-2)$, $(a-1, a-2, a-2, b)$, and $(a-1, a-1, b-1, b-1)$. Solutions of the first kind exist if and only if $b \geq 3$. Under this constraint, a solution can be realized by $\mu = 2^{b-2} - 1, \nu = 2^a + 2^{b-1} + 2^{b-2} - 1$ and $\mu = 2^{b-1} + 2^{b-2} - 1, \nu = 2^a + 2^{b-2} - 1$. In each case, we find that $f_\mu f_\nu \equiv 4\,(8)$; thus, the total contribution of these terms to the sum vanishes modulo 8. Solutions of the second kind different from solutions of the third kind can be realized if and only if $a \geq b+3$, and the same argument as for solutions of the first kind shows that their contribution to the sum vanishes modulo 8. Finally, solutions of the third kind can only be realized as $\mu = \nu = 2^{a-1} + 2^{b-1} - 1$, contradicting the summation condition $\mu < \nu$. Summarizing, we find that summands corresponding to pairs (μ, ν) with $\mathfrak{s}_2(\nu+1) + \mathfrak{s}_2(\mu+1) = 4$ can be neglected altogether.

Terms corresponding to pairs (μ, ν) with $\mathfrak{s}_2(\nu+1)+\mathfrak{s}_2(\mu+1) = 3$ are represented by solutions of the equation $2^x + 2^y + 2^z = 2^a + 2^b$. This equation has the solutions $(a, b-1, b-1)$ and $(a-1, a-1, b)$. Solutions of the first kind can be realized if and only if $b \geq 2$, in which case we have $f_\mu f_\nu \equiv 6\,(8)$; whereas solutions of the second kind exist if and only if $a \geq b+2$; we have $f_\mu f_\nu \equiv 6\,(8)$ if $a \geq b+3$, and $f_\mu f_\nu \equiv 2\,(8)$ if $a = b+2$. Note that, for $\lambda > 20$, the conditions $a \leq b+2$ and $b \leq 2$ are mutually exclusive; hence, the contribution of these pairs modulo 8 to the right-hand side sum in (3) is 6, if $b = 1$ or $a = b+1$; 0, if $b \geq 2$ and $a = b+2$; and 4, if $b \geq 2$ and $a \geq b+3$.

Finally, pairs (μ, ν) with $\mathfrak{s}_2(\mu+1) + \mathfrak{s}_2(\nu+1) = 2$ correspond to solutions of the equation $2^x + 2^y = 2^a + 2^b$, which are uniquely given by $\mu = 2^b - 1, \nu = 2^a - 1$. The contribution corresponding to this pair is $1\,(8)$ for $b \neq 2$, and $5\,(8)$ for $b = 2$. We obtain the congruence modulo 16

$$f_\lambda \equiv \underbrace{\left\{\begin{matrix} 4, & b \geq 2 \\ 0, & b = 1 \end{matrix}\right\} + \left\{\begin{matrix} 12, & b \geq 2 \\ 0, & b = 1 \end{matrix}\right\}}_{\equiv 0} + \left\{\begin{matrix} 12, & a \geq b+3 \\ 4, & a = b+2 \\ 0, & a = b+1 \end{matrix}\right\} + \left\{\begin{matrix} 2, & b \neq 2 \\ 10, & b = 2 \end{matrix}\right\},$$

which implies (iii).

(iv) Write $\lambda = 2^a + 2^b + 2^c - 1$. By Lemma 2, we may assume that $3 \leq \mathfrak{s}_2(\mu+1) + \mathfrak{s}_2(\nu+1) \leq 4$. The case $\mathfrak{s}_2(\mu+1) + \mathfrak{s}_2(\nu+1) = 4$ corresponds to solutions of the equation $2^x + 2^y + 2^z + 2^u = 2^a + 2^b + 2^c$. All solutions consist of one repeated value, which is one of $a-1, b-1, c-1$, and two values equal to the other two-parameters. The repeated value has to occur in both μ and ν, and the largest nonrepeated value has to occur in ν, since we assume $\mu < \nu$. The smallest nonrepeated value can occur in either μ or ν; thus, the number of realizations is always even. Since $f_\mu f_\nu \equiv 4\,(8)$ in each case, we find that the contribution of all these solutions vanishes modulo 8.

Pairs (μ, ν) satisfying $\mathfrak{s}_2(\mu+1) + \mathfrak{s}_2(\nu+1) = 3$ correspond to solutions of the equation $2^x + 2^y + 2^z = 2^a + 2^b + 2^c$, which are unique, and can be realized as $\mu = 2^c - 1, \nu = 2^a + 2^b - 1$; $\mu = 2^b - 1, \nu = 2^a + 2^c - 1$; and $\mu = 2^b + 2^c - 1, \nu = 2^a - 1$. In the first case, we have the following congruences mod 8:

$$f_\mu \equiv \begin{cases} 1, & c = 2 \\ 5, & c \neq 2, \end{cases} \qquad f_\nu \equiv \begin{cases} 2, & a = b+1 \\ 6, & a \geq b+2, \end{cases}$$

and therefore

$$f_\mu f_\nu \equiv \begin{cases} 2, & a = b+1 \\ 6, & a \geq b+2. \end{cases} \pmod 8$$

In the second case, we always have $f_\mu f_\nu \equiv 6 \pmod 8$, and in the third case

$$f_\mu f_\nu \equiv \begin{cases} 2, & b = c+1 \\ 6, & b \geq c+2. \end{cases} \pmod 8$$

Adding up these contributions, we find that the sum is congruent to $2 + 4k$, where $0 \leq k \leq 2$ is the number of equations among $a = b+1$, $b = c+1$ holding true.

If $c \neq 1$, then $(\lambda-1)/2$ is odd and satisfies $\mathfrak{s}_2((\lambda-1)/2) = 3$; hence, $f_{(\lambda-1)/2}$ is divisible by 4, and $(f_{(\lambda-1)/2})^2$ does not contribute modulo 16. If $c = 1$, then $(\lambda-1)/2$ is even and at least 10; hence, $(f_{(\lambda-1)/2})^2$ does not contribute to this case either. Thus, in each case, we obtain assertion (iv).

(v) Write $\lambda = 2^a + 2^b + 2^c + 2^d - 1$. If $\mathfrak{s}_2(\lambda+1) = 4$, then either $(\lambda-1)/2$ is even, or $\mathfrak{s}_2(\lambda-1)/2) = 4$; thus, in any case, $f_{(\lambda-1)/2}$ is divisible by 8. Solutions of the equation $2^x + 2^y + 2^z + 2^u = 2^a + 2^b + 2^c + 2^d$ are unique up to permutation of variables, and for every realization we have $f_\mu f_\nu \equiv 4 \pmod 8$. These realizations correspond to nonempty subsets of $\{b, c, d\}$, hence there are 7 of them, and the sum is congruent to 4 modulo 8.

(vi) If $\mathfrak{s}_2(\lambda+1) \geq 5$, then Lemma 2 gives that every solution $\mu + \nu = \lambda - 1$ contributes 0 modulo 16; similarly, $f_{(\lambda-1)/2}$ is divisible by 8, and we obtain $f_\lambda \equiv 0\,(16)$.

(i), final version: We now use parts (ii)–(vi) to prove (i) modulo 64 by computing the right-hand side of (2). Assume first that $\lambda = 2^a + 4$. Then $f_{\lambda-5}$ is odd, $f_{\lambda-1}, f_{\lambda-3} \equiv 6\,(8)$, and $\lambda \equiv 4\,(16)$. We deduce that

$$f_\lambda \equiv 6 \cdot 4 \cdot 6 + 56 \cdot 6 + 32 \equiv 0 \pmod{64}.$$

In all other cases, $f_{\lambda-5}$ is even, and we obtain the relation

$$f_\lambda \equiv 6\lambda f_{\lambda-1} + 56 f_{\lambda-3} \pmod{64}. \tag{4}$$

If λ is a power of 2, then $\mathfrak{s}_2(\lambda-2) \geq 5$, and our claim follows in this case. If λ is divisible by 16, but not a power of 2, $f_{\lambda-1}$ is even, and $\mathfrak{s}_2(f_{\lambda-3}) \geq 4$, and we deduce $f_\lambda \equiv 0\,(64)$ again. If $\lambda \equiv 8\,(16)$, then $\mathfrak{s}_2(\lambda) = \mathfrak{s}_2(\lambda-2) - 1$, and both summands on the right-hand side of (4) are divisible by the same power of 2, which is at least 32, hence, their sum is divisible by 64. The same argument applies if $\lambda \equiv 4\,(8)$.

We will finish the proof by inspecting the remaining residue classes modulo 16 one after the other. Note that in each case we may assume that $\mathfrak{s}_2(\lambda) \leq 4$, since

$\mathfrak{s}_2(\lambda-2) \geq \mathfrak{s}_2(\lambda)-1$. Moreover, if $\mathfrak{s}_2(\lambda)=4$, then $\mathfrak{s}_2(\lambda)=3$, and both summands are divisible by 32, but not by 64, implying $f_\lambda \equiv 0\,(64)$.

If $\lambda \equiv 2\,(16)$ and $\mathfrak{s}_2(\lambda)=2$, then we have $f_{\lambda-1} \equiv 14\,(16)$ and $f_{\lambda-3} \equiv 5\,(8)$, from which we obtain that $f_\lambda \equiv 12 \cdot 14 + 56 \cdot 5 \equiv 0\,(64)$. If $\mathfrak{s}_2(\lambda)=3$, write $\lambda = 2^a + 2^b + 2$ with $a > b \geq 4$. If $a = b+1$, then $f_{\lambda-1} \equiv 12\,(16)$ and $f_{\lambda-3} \equiv 2\,(8)$, hence, $f_\lambda \equiv 12\cdot12+56\cdot2 \equiv 0\,(64)$. If $a=b+2$, then $f_{\lambda-1} \equiv 4\,(16)$ and $f_{\lambda-3} \equiv 6\,(8)$, thus $f_\lambda \equiv 12 \cdot 4 + 56 \cdot 6 \equiv 0\,(64)$. If $a \geq b+3$, then $f_{\lambda-1} \equiv 4\,(16)$ and $f_{\lambda-3} \equiv 6\,(8)$, and $f_\lambda \equiv 0\,(64)$ as before.

If $\lambda \equiv 6\,(16)$ and $\mathfrak{s}_2(\lambda)=3$, then $\lambda = 2^a + 6$. Hence, $f_{\lambda-1} \equiv 12\,(16)$ and $f_{\lambda-3} \equiv 6\,(8)$, and we obtain $f_\lambda \equiv 36 \cdot 12 + 56 \cdot 6 \equiv 0\,(64)$.

If $\lambda \equiv 10\,(16)$ and $\mathfrak{s}_2(\lambda)=3$, then $\lambda = 2^a+10$. Checking the case $\lambda=26$ numerically, we may assume $a \geq 5$; therefore $f_{\lambda-1} \equiv 4\,(16)$ and $f_{\lambda-3} \equiv 6\,(8)$, and hence, $f_\lambda \equiv 60 \cdot 4 + 56 \cdot 6 \equiv 0\,(64)$.

Finally, if $\lambda \equiv 14\,(16)$, we have $\mathfrak{s}_2(\lambda) \geq 4$, a case which has already been dealt with. □

Note that computing the values $f_\lambda \bmod 2^k$ allows one to decide whether f_λ, λ even, is divisible by 2^{k+2}, and this in turn allows one to check whether f_λ, λ odd, is divisible by 2^{k+3}.

3. Proof of Proposition 1

We begin by proving the statement for f_λ; the assertion for s_n is then deduced by the connection between these two functions established in [6]. We prove our claim on f_λ by induction on λ. Write the recursion formula (1) in the form

$$f_\lambda = 6\lambda f_{\lambda-1} + f_{(\lambda-1)/2}^2 + 2 \sum_{1 \leq \mu < (\lambda-1)/2} f_\mu f_{\lambda-\mu-1}, \tag{5}$$

where $f_{(\lambda-1)/2}$ is to be interpreted as zero if λ is even. Estimating the 2-part of f_λ by the minimum taken over the right-hand terms, we obtain

$$\nu_2(f_\lambda) \geq \min_{1 \leq \mu < \frac{\lambda-1}{2}} \Big(1+\nu_2(\lambda)+\nu_2(f_{\lambda-1}), 2\nu_2(f_{(\lambda-1)/2}), 1+\nu_2(f_\mu)+\nu_2(f_{\lambda-\mu-1})\Big).$$

Suppose first that λ is even. Then, by the induction hypothesis,

$$\begin{aligned} \nu_2(f_{\lambda-1}) &\geq \mathfrak{s}_2(\lambda)-1 \\ &= \mathfrak{s}_2(\lambda+1)-2, \\ \nu_2(f_\mu)+\nu_2(f_{\lambda-\mu-1}) &\geq \mathfrak{s}_2(\mu+1)+\mathfrak{s}_2(\lambda-\mu)-1, \end{aligned}$$

since at least one of $\mu+1$ and $\lambda-\mu$ is even. From these estimates our claim follows for λ even. For λ odd, the first term on the right-hand side of (5) is divisible by $2^{s_2(\lambda+1)}$, since $\lambda-1$ is odd. The 2-part of the second term can be computed as

$$2\nu_2(f_{(\lambda-1)/2}) \geq 2(\mathfrak{s}_2(\lambda+1)-1) \geq \mathfrak{s}_2(\lambda+1),$$

unless $\lambda+1$ is a power of 2; however, in this case f_λ is odd and our claim holds true anyway. Moreover, the same computation as in the case λ even reveals that every single term is divisible by $2^{\mathfrak{s}_2(\lambda+1)-2}$ and occurs twice. Hence, it suffices to count the number of terms with the property $\nu_2(f_\mu)+\nu_2(f_{\lambda-\mu-1}) = \mathfrak{s}_2(\lambda+1)-2$.

By the claim for even λ we find that we may assume μ odd, and by our induction hypothesis we find for such values

$$\nu_2(f_\mu) + \nu_2(f_{\lambda-\mu-1}) = \mathfrak{s}_2(\mu+1) + \mathfrak{s}_2(\lambda-\mu) - 2.$$

The right-hand side is equal to $\mathfrak{s}_2(\lambda+1)$ if and only if there are no carries in the addition $(\mu+1)+(\lambda-\mu)$, which is equivalent to the statement that the places at which the digital representation of $\mu+1$ has the digit 1 are among the places where λ has the digit 1. Thus, we find

$$f_\lambda \equiv 2^{\mathfrak{s}_2(\lambda+1)-2}N \pmod{2^{\mathfrak{s}_2(\lambda-\mu)}},$$

where N denotes the number of integers μ satisfying the following conditions:

(i) $1 \leq \mu \leq \lambda - 2$,
(ii) μ is odd,
(iii) $\mu+1$ has 1's only at places where $\lambda+1$ has 1's.

Since $\lambda+1$ is even, $\mu+1$ cannot have a 1 as its last digit, that is, the last condition already implies the second one. Clearly, the number of integers satisfying the last condition equals the number of subsets of those places, which is $2^{\mathfrak{s}_2(\lambda+1)}$, and it remains to check how many of these satisfy $1 \leq \mu \leq \lambda - 2$. The integers $\mu = -1$ and $\mu = \lambda$ satisfy the last condition, but fall out of the range; $\mu = 0$ and $\mu = \lambda - 1$ are even and therefore cannot satisfy the last condition, while all other integers are within the range for μ. Hence, $N = 2^{\mathfrak{s}_2(\lambda-\mu)} - 2$, and we obtain

$$f_\lambda \equiv 2^{\mathfrak{s}_2(\lambda+1)-1} \pmod{2^{\mathfrak{s}_2(\lambda-\mu)}},$$

which is equivalent to our claim.

In order to prove part (ii) of the proposition, we have to express s_n in terms of the function f_λ. This is done in the next lemma.

Lemma 5. *We have*

$$s_n(\Gamma) = \sum_{m=(n-2)/4}^{n/3} \sum_{\nu=0}^{(4m-n+2)/2} \frac{n(m-2\nu+2)!}{3^m(n-3m)!(4m-n-2\nu+2)!} N(\nu, m),$$

where

$$N(\nu,m) = \frac{3^{m-1}2^{m-2\nu+1}}{(m-2\nu+2)!} f_{\nu-1} \prod_{h=0}^{m-2\nu} (2m-\nu-2h-1), \quad (\nu \geq 2, m \geq 2\nu - 1),$$

and

$$N(0,m) = \frac{3^m(2m)!}{m!(m+2)!},\ N(1,m) = \frac{12^m}{4m},\ N(\nu, 2\nu-2) = \frac{3^{2\nu-2}(n-6\nu+6)!^2}{n} f_{\nu-1}.$$

Proof. This is essentially the content of Theorem 1.5 and Propositions 1.7–1.9 in [6]. □

Lemma 6. *If $\nu \geq 3$ is odd, and $m \geq 2\nu - 1$, we have*

$$\nu_2(N(\nu,m)) \geq \nu_2(f_{\nu-1}) + (m - 2\nu + 1) + \mathfrak{s}_2(m - 2\nu + 2) - 1,$$

whereas for $\nu \geq 2$ even and $m \geq 2\nu - 1$,

$$\nu_2(N(\nu,m) = \nu_2(f_{\nu-1}) + \mathfrak{s}_2(m-2\nu+2) - 1.$$

Moreover,

$$\nu_2(N(0,m)) = \mathfrak{s}_2(m+2) - 2 \text{ and } \nu_2(N(1,m)) = 2m - 2 - \nu_2(m).$$

Proof. This follows from Lemma 5 using the formula $\nu_2(n!) = n - \mathfrak{s}_2(n)$ due to Legendre. □

Set

$$c(\nu, m) = \frac{n(m-2\nu+2)!}{3^m(n-3m)!(4m-n-2\nu+2)!} N(\nu, m),$$

and split s_n into the sums

$$\begin{aligned} \sum\nolimits_0 &= \sum_{m=(n-2)/4}^{n/3} c(0,m) \\ \sum\nolimits_1 &= \sum_{m=(n-2)/4}^{n/3} c(1,m) \\ \sum\nolimits_2 &= \sum_{m=(n-2)/4}^{n/3} \sum_{\nu \geq 2} c(\nu,m). \end{aligned}$$

Next we check that every single summand is divisible by the claimed power of 2. We have

$$\nu_2(c(0,m)) = \mathfrak{s}_2(m+2) - 4 + \nu_2(n) + \mathfrak{s}_2(n-3m) + \mathfrak{s}_2(4m-n+2).$$

Since $n + 6 = 4(n-3m) + 3(4m-n+2)$, we have

$$\mathfrak{s}_2(n+6) \leq \mathfrak{s}_2(4(n-3m)) + \mathfrak{s}_2(3(4m-n+2)) \leq s_2(n-3m) + 2s_2(4m-n+2),$$

and therefore

$$\begin{aligned} \nu_2(c(0,m)) &\geq \frac{\mathfrak{s}_2(n+6) + \mathfrak{s}_2(n-3m) + 2s_2(m+2) - 8}{2} + \nu_2(n) \\ &\geq \frac{1}{2}\mathfrak{s}_2(n+6) + \nu_2(n) - \frac{5}{2}. \end{aligned}$$

For $\sum_1$ we have

$$\nu_2(c(1,m)) \geq 2m - 2 - \nu_2(m) \geq 2m - 2 - \frac{\log m}{\log 2} \geq \frac{n-6}{2} - \frac{\log n}{\log 2}.$$

Since $\mathfrak{s}_2(n+6) \leq \frac{\log(n+6)}{\log 2} + 1$, our claim follows provided that

$$n > 2 + 4\frac{\log n}{\log 2} + \frac{\log(n+6)}{\log 2},$$

which is satisfied for $n \geq 26$. Finally, we consider $\sum_2$. For ν odd, each factor in the product defining $N(\nu, m)$ is even, and we obtain

$$\begin{aligned} \nu_2(c(\nu,m)) &\geq \nu_2(n) + \mathfrak{s}_2(n-3m) + \mathfrak{s}_2(4m-n-2\nu+2) + m - 2\nu + \nu_2(f_{\nu-1}) \\ &\geq \frac{1}{2}\mathfrak{s}_2(n+6) + \nu_2(n) + m - 2\nu + 1. \end{aligned}$$

If ν is even, the product in the definition of $N(\nu, m)$ is odd, and we obtain

$$\begin{aligned}\nu_2(c(\nu,m)) &= \nu_2(n) + \mathfrak{s}_2(n-3m) + \mathfrak{s}_2(4m-n-2\nu+2) + \nu_2(f_{\nu-1}) - 1\\ &\geq \frac{1}{2}\mathfrak{s}_2(n+6) + \nu_2(n).\end{aligned}$$

Hence, for $n \geq 26$ each $c(\nu, m)$ contributing to s_n in the above formula is divisible by a sufficiently large power of 2, which proves our claim for $n \geq 26$. For smaller values of n the 2-part of s_n is easily computed from the recurrence relation (6) for s_n, or read off Newman's table in [5], and our claim turns out to hold in these cases as well.

4. Proof of Theorem 2

The proof of Theorem 2 is based on a recurrence relation first established by Godsil, Imrich, and Razen [1], who used it to give an alternative proof of Stothers' result concerning the parity of s_n.

Lemma 7. *For $n \geq 10$, we have*

$$\begin{aligned}s_n = 4s_{n-3} + 2s_{n-4} + (n-3)s_{n-6} - 2s_{n-7} - (n-6)s_{n-9}\\ + \sum_{i=1}^{n-7} s_i s_{n-i-6} - \sum_{i=1}^{n-10} s_i s_{n-i-9}.\end{aligned} \tag{6}$$

This is [1, Theorem 1].

We prove Theorem 2 by induction on n. Suppose that our claim holds true for $n-9, \dots, n-1$, and that $n > 30$. We begin by evaluating the linear terms in (6) using the inductive hypothesis. We have

$$\begin{aligned}4s_{n-3} &\equiv \{4|n = 2^a, 2^a-3, a \geq 3\}\\ 2s_{n-4} &\equiv \{2|n = 2^a-2, 2^a+1, a \geq 3\}\\ &\quad + \{4|n = 2^a+2^b+1, 2^a+2^b-2, a > b \geq 2\}\\ (n-3)s_{n-6} &\equiv \{1|n = 2^a, a \geq 3\} + \{4|n = 3\cdot 2^a, a \geq 3\}\\ &\quad + \{6|n = 2^a+2^b, a > b \geq 2\} + \{4|n = 2^a+9, a \geq 3\}\\ &\quad + \{4|n = 2^a+2^b+2^c, a > b > c \geq 2\}\\ -2s_{n-7} &\equiv \{6|n = 2^a+1, 2^a+4, a \geq 3\}\\ &\quad + \{4|n = 2^a+2^b+4, 2^a+2^b+1, a > b \geq 2\}\\ -(n-6)s_{n-9} &\equiv \{7|n = 2^a+3, a \geq 3\} + \{4|n = 3\cdot 2^a+3, a \geq 3\}\\ &\quad + \{2|n = 2^a+2^b+3, a > b \geq 2\} + \{4|n = 2^a+12\}\\ &\quad + \{4|n = 2^a+2^b+2^c+3, a > b > c \geq 2\}.\end{aligned}$$

In the nonlinear parts, each summand either occurs twice or is squared; hence, to determine the behaviour of these sums modulo 8, it suffices to know the occurring values s_ν mod 4. One easily checks that Theorem 2 holds true modulo 4 for $n \leq 21$, that is, in what follows we may use the inductive hypothesis even in the range where we know it not to hold modulo 8.[1] For each n, precisely one term occurs only once, namely $s^2_{(n-6)/2}$ or $s^2_{(n-9)/2}$, depending on whether n is even or odd. The contribution of this term is congruent to

$$\begin{aligned}&\{1|n=2^a-6,2^a\}+\{7|n=2^a-3,2^a+3\}+\\&\{4|n=2^a+2^b,2^a+2^b+3,2^a+2^b-6,2^a+2^b-3,2^a+12,2^a+15,a>b\geq 2\}.\end{aligned}$$

In the remaining sum we may neglect all terms with both factors even as well as all terms with one factor divisible by 4. On noting the different signs of the sums we find the contribution of these sums to be congruent to

$$\begin{aligned}&2\#\{n-6=2^a-3+2^b-3,a>b\geq 2\}\\&+2\#\{n-6=2^a-6+2^b-3,a\geq 3,b\geq 2\}\\&+2\#\{n-6=2^a-6+2^b-6,a>b\geq 3\}\\&-2\#\{n-9=2^a-3+2^b-3,a>b\}-2\#\{n-9=2^a-6+2^b-3,a\geq 3,b\geq 2\}\\&-2\#\{n-9=2^a-6+2^b-6,a>b\geq 3\}\\&+\{4|n-6=2^a-3+2^b+2^c-3,b>c\geq 2,a\geq 2,b\geq 4\}\\&+\{4|n-6=2^a-6+2^b+2^c-3,b>c\geq 2,a\geq 3,b\geq 4\}\\&+\{4|n-6=2^a-3+2^b+2^c-6,b>c\geq 2,a\geq 2\}\\&+\{4|n-6=2^a-6+2^b+2^c-6,b>c\geq 2,a\geq 3\}\\&+\{4|n-9=2^a-3+2^b+2^c-3,b>c\geq 2,a\geq 2,b\geq 4\}\\&+\{4|n-9=2^a-6+2^b+2^c-3,b>c\geq 2,a\geq 3,b\geq 4\}\\&+\{4|n-9=2^a-3+2^b+2^c-6,b>c\geq 2,a\geq 2\}\\&+\{4|n-9=2^a-6+2^b+2^c-6,b>c\geq 2,a\geq 3\}\\&+\{4|n-6=2^a-3+2^b+3,a,b\geq 2\}+\{4|n-6=2^a-6+2^b+3,a\geq 3,b\geq 2\}\\&+\{4|n-9=2^a-3+2^b+3,a,b\geq 2\}+\{4|n-9=2^a-6+2^b+3,a\geq 3,b\geq 2\}.\end{aligned}$$

Due to the fact that among the conditions describing $s_n \bmod 4$ there are several pairs differing by 3, several terms occur twice in the last sum. Moreover, the contributions in line 5 and 10 coincide up to the condition on a; hence, their sum is equal modulo 8 to the corresponding expression with $a=2$. In what follows, we will repeatedly use the assumption $n>30$ to ignore terms like

$$\{4|n=2^a+3,a\geq 4\}+\{4|n=2^a+3,a\geq 3\}.$$

[1] This phenomenon is one of the reasons why we believe s_n to behave regularly modulo higher powers of 2 as well. Moreover, the prime number 2 is clearly distinguished here, and we do not expect any similar regular patterns for other primes.

In this way we may simplify the last displayed expression as follows:

$$\begin{aligned}
&2\#\{n=2^a+2^b, a>b\geq 2\}+2\#\{n=2^a+2^b-3, a\geq 3, b\geq 2\}\\
&+2\#\{n=2^a+2^b-6, a>b\geq 3\}-2\#\{n=2^a+2^b+3, a>b\}\\
&-2\#\{n=2^a+2^b, a\geq 3, b\geq 2\}-2\#\{n=2^a+2^b-3, a>b\geq 3\}\\
&+4\#\{n=2^b+2^c+4, b>c\geq 2, b\geq 4\}+4\#\{n=2^a+9, a\geq 3\}\\
&+4\#\{n=2^b+2^c+1, b>c\geq 2\}+4\#\{n=2^a+2^b+2^c-6, b>c\geq 2, a\geq 3\}\\
&+4\#\{n=2^a+2^b+2^c+3, b>c\geq 2, a\geq 2, b\geq 4\}\\
&+4\#\{n=2^a+2^b+2^c, b>c\geq 2, a\geq 2\}\\
&+4\#\{n=2^a+2^b+2^c-3, b>c\geq 2, a\geq 3\}\\
&+4\#\{n=2^a+2^b+3, a\geq 3, b\geq 2\}+4\#\{n=2^a+2^b+9, a, b\geq 2\}.
\end{aligned}$$

Next consider for example the quantity $4\#\{n=2^a+2^b+6, a\geq 3, b\geq 2\}$. If (a,b) is a solution with $a>b\geq 3$, then (b,a) is also a solution, that is, the number of solutions is even, unless n is of the form $n=2^a+10, a\geq 3$, or n is of the form 2^a+6 with $a\geq 4$. The same argument may be applied to several other terms as well, which allows us to simplify the expression further to obtain the following:

$$\begin{aligned}
&2\#\{n=2^a+2^b, a>b\}+4\#\{n=2^a+1, a\geq 3\}+2\#\{n=2^a-3, a\geq 4\}\}\\
&+4\#\{n=2^a+2^b-3, a>b\geq 2\}+2\#\{n=2^a+2^b-6, a>b\geq 3\}\\
&-2\#\{n=2^a+2^b+3, a>b\}-2\#\{n=2^a+4, a\geq 3\}-2\#\{n=2^a, a\geq 4\}\\
&+4\#\{n=2^a+2^b, a>b\geq 2\}-2\#\{n=2^a+2^b-3, a>b\geq 3\}\\
&+4\#\{n=2^b+2^c+4, b>c\geq 2, b\geq 4\}+4\#\{n=2^b+2^c+1, b>c\geq 2\}\\
&+4\#\{n=2^a+2^b+2^c-6, b>c\geq 2, a\geq 3\}\\
&+4\#\{n=2^a+2^b+2^c+3, b>c\geq 2, a\geq 2, b\geq 4\}\\
&+4\#\{n=2^a+2^b+2^c, b>c\geq 2, a\geq 2\}\\
&+4\#\{n=2^a+2^b+2^c-3, b>c\geq 2, a\geq 3\}\\
&+4\#\{n=2^a+7, a\geq 3\}+4\#\{n=2^a+3, a\geq 4\}.
\end{aligned}$$

Finally, consider the quantity

$$\#\{n=2^a+2^b+2^c, b>c\geq 2, a\geq 2\}.$$

Let (a,b,c) be a solution counted. If all three components are distinct, there are no solutions with two of the variables equal to each other, and there are 3 possible orderings of a,b and c. In this case, we deduce

$$\begin{aligned}
&\#\{n=2^a+2^b+2^c, b>c\geq 2, a\geq 2\}\\
&\qquad\qquad\equiv\#\{n=2^a+2^b+2^c, a>b>c\geq 2\}\pmod 2.
\end{aligned}$$

If two variables are equal, then either n is of the form $n=2^x+2^y$ with $x\neq y$, or n is a power of two. In the former case, we either have $a=b=x-1$, $c=y$, or $a=c=y-1$, $b=x$. The latter is possible only for $y\geq 3$, while the former is possible only for $x \geq y+2$, hence, in this case the number of solutions is congruent modulo 2 to

$$\#\{n = 2^a + 4, a \geq 4\} + \#\{n = 3 \cdot 2^a, a \geq 4\}.$$

If $n = 2^x$ is a power of two, we necessarily have $a = c = 2^{x-2}$, $b = 2^{x-1}$ and there is precisely one solution. Thus, we deduce that

$$\#\{n = 2^a + 2^b + 2^c, b > c \geq 2, a \geq 2\}$$

is congruent modulo 2 to

$$\begin{aligned}&\#\{n = 2^a + 2^b + 2^c, a > b > c \geq 2\} + \#\{n = 2^a + 4, a \geq 4\}\\ &+ \#\{n = 2^a, a \geq 4\} + \#\{n = 3 \cdot 2^a, a \geq 4\}.\end{aligned}$$

In this way we may treat the other terms and obtain

$$\begin{aligned}&2\#\{n = 2^a + 2^b, a > b\} + 4\#\{n = 2^a + 1, a \geq 3\} + 2\#\{n = 2^a - 3, a \geq 4\}\\
&+ 4\#\{n = 2^a + 2^b - 3, a > b \geq 2\} + 2\#\{n = 2^a + 2^b - 6, a > b \geq 3\}\\
&- 2\#\{n = 2^a + 2^b + 3, a > b\} - 2\#\{n = 2^a + 4, a \geq 3\} - 2\#\{n = 2^a, a \geq 4\}\\
&+ 4\#\{n = 2^a + 2^b, a > b \geq 2\} - 2\#\{n = 2^a + 2^b - 3, a > b \geq 3\}\\
&+ 4\#\{n = 2^b + 2^c + 4, b > c \geq 2, b \geq 4\} + 4\#\{n = 2^b + 2^c + 1, b > c \geq 2\}\\
&+ 4\#\{n = 2^b + 2^c - 2, b > c \geq 2\} + 4\#\{n = 2^a + 2^b + 2^c - 6, a > b > c \geq 2\}\\
&+ 4\#\{n = 2^a - 2, a \geq 5\} + 4\#\{n = 2^a - 6, a \geq 5\} + 4\#\{n = 3 \cdot 2^a - 6, a \geq 5\}\\
&+ 4\#\{n = 2^a + 15, a \geq 2\} + 4\#\{n = 2^a + 2^b + 2^c + 3, a > b > c \geq 2\}\\
&+ 4\#\{n = 2^a + 7, a \geq 4\} + 4\#\{n = 2^a + 3, a \geq 4\}\\
&+ 4\#\{n = 3 \cdot 2^a + 3, a \geq 4\} + 4\#\{n = 2^a + 2^b + 2^c, a > b > c \geq 2\}\\
&+ 4\#\{n = 2^a + 4, a \geq 4\} + 4\#\{n = 2^a, a \geq 4\} + 4\#\{n = 3 \cdot 2^a, a \geq 4\}\\
&+ 4\#\{n = 2^b + 2^c + 1, b > c \geq 2\} + 4\#\{n = 2^a + 2^b + 2^c - 3, a > b > c \geq 2\}\\
&+ 4\#\{n = 2^a + 1, a \geq 4\} + 4\#\{n = 2^a - 3, a \geq 4\} + 4\#\{n = 3 \cdot 2^a - 3, a \geq 4\}\\
&+ 4\#\{n = 2^a + 7, a \geq 3\} + 4\#\{n = 2^a + 3, a \geq 4\}.\end{aligned}$$

It follows from the uniqueness of the dyadic representation, that each of the sets in the last expression is either empty or contains precisely one element. Hence, we can write e.g., $4\#\{n = 2^a + 10, a \geq 3\}$ as $\{4 | n = 2^a + 10, a \geq 3\}$, and combine the last result with our previous computations. Collecting terms we find that, modulo 8, s_n is congruent to

$$\begin{aligned}&\{4|n = 2^a, 2^a - 3, a \geq 3\} + \{2|n = 2^a - 2, 2^a + 1, a \geq 3\}\\
&+ \{4|n = 2^a + 2^b + 1, 2^a + 2^b - 2, a > b \geq 2\} + \{3|n = 2^a, a \geq 3\}\\
&+ \{4|n = 3 \cdot 2^a, a \geq 3\} + \{6|n = 2^a + 2^b, a > b \geq 2\} + \{4|n = 2^a + 9, a \geq 3\}\\
&+ \{4|n = 2^a + 2^b + 2^c, a > b > c \geq 2\} + \{6|n = 2^a + 1, 2^a + 4, a \geq 3\}\\
&+ \{4|n = 2^a + 2^b + 4, 2^a + 2^b + 1, a > b \geq 2\} + \{7|n = 2^a + 3, a \geq 3\}\\
&+ \{4|n = 3 \cdot 2^a + 3, a \geq 3\} + \{2|n = 2^a + 2^b + 3, a > b \geq 2\} + \{4|n = 2^a + 12\}\\
&+ \{4|n = 2^a + 2^b + 2^c + 3, a > b > c \geq 2\} + \{1|n = 2^a - 6, 2^a\}\end{aligned}$$

$$
\begin{aligned}
&+\{7|n=2^a-3,2^a+3\}+\{2|n=2^a+2^b,a>b\}+\{4|n=2^a+1,a\geq 3\}\\
&+\{4|n=2^a+2^b,2^a+2^b+3,2^a+2^b-6,2^a+2^b-3,a>b\geq 2\}\\
&+\{4|n=2^a+12,2^a+15,a\geq 2\}+\{2|n=2^a-3,a\geq 4\}\}\\
&+\{4|n=2^a+2^b-3,a>b\geq 2\}+\{2|n=2^a+2^b-6,a>b\geq 3\}\\
&-\{2|n=2^a+2^b+3,a>b\}-\{2|n=2^a+4,a\geq 3\}-\{2|n=2^a,a\geq 4\}\\
&+\{4|n=2^a+2^b,a>b\geq 2\}-\{2|n=2^a+2^b-3,a>b\geq 3\}\\
&+\{4|n=2^b+2^c+4,b>c\geq 2,b\geq 4\}+\{4|n=2^b+2^c+1,b>c\geq 2\}\\
&+\{4|n=2^b+2^c-2,b>c\geq 2\}+\{4|n=2^a+2^b+2^c-6,a>b>c\geq 2\}\\
&+\{4|n=2^a-2,a\geq 5\}+\{4|n=2^a-6,a\geq 5\}+\{4|n=3\cdot 2^a-6,a\geq 5\}\\
&+\{4|n=2^a+15,a\geq 2\}+\{4|n=2^a+2^b+2^c+3,a>b>c\geq 2\}\\
&+\{4|n=2^a+7,a\geq 4\}+\{4|n=2^a+3,a\geq 4\}+\{4|n=3\cdot 2^a+3,a\geq 4\}\\
&+\{4|n=2^a+2^b+2^c,a>b>c\geq 2\}+\{4|n=2^a+4,a\geq 4\}+\{4|n=2^a,a\geq 4\}\\
&+\{4|n=3\cdot 2^a,a\geq 4\}+\{4|n=2^b+2^c+1,b>c\geq 2\}\\
&+\{4|n=2^a+2^b+2^c-3,a>b>c\geq 2\}+\{4|n=2^a+1,a\geq 4\}\\
&+\{4|n=2^a-3,a\geq 4\}+\{4|n=3\cdot 2^a-3,a\geq 4\}\\
&+\{4|n=2^a+7,a\geq 3\}+\{4|n=2^a+3,a\geq 4\}.
\end{aligned}
$$

To ease further computations, we consider sets with one, two, and three parameters separately. Sets defined by one-parameter contribute

$$
\begin{aligned}
&\{4|n=2^a,2^a-3,a\geq 3\}+\{2|n=2^a-2,2^a+1,a\geq 3\}+\{1|n=2^a,a\geq 3\}\\
&+\{4|n=3\cdot 2^a,a\geq 3\}+\{4|n=2^a+9,a\geq 3\}+\{6|n=2^a+1,2^a+4,a\geq 3\}\\
&+\{7|n=2^a+3,a\geq 3\}+\{4|n=3\cdot 2^a+3,a\geq 3\}+\{4|n=2^a+12\}\\
&+\{1|n=2^a-6,2^a\}+\{7|n=2^a-3,2^a+3\}\\
&+\{4|n=2^a+12,2^a+15,a>b\geq 2\}+\{4|n=2^a+1,a\geq 3\}\\
&+\{2|n=2^a-3,a\geq 4\}\}-\{2|n=2^a+4,a\geq 3\}\\
&-\{2|n=2^a,a\geq 4\}+\{4|n=2^a-2,a\geq 5\}+\{4|n=2^a-6,a\geq 5\}\\
&+\{4|n=3\cdot 2^a-6,a\geq 5\}+\{4|n=2^a+15,a\geq 2\}+\{4|n=2^a+7,a\geq 4\}\\
&+\{4|n=2^a+3,a\geq 4\}+\{4|n=3\cdot 2^a+3,a\geq 4\}+\{4|n=2^a+4,a\geq 4\}\\
&+\{4|n=2^a,a\geq 4\}+\{4|n=3\cdot 2^a,a\geq 4\}+\{4|n=2^a+1,a\geq 4\}\\
&+\{4|n=2^a-3,a\geq 4\}+\{4|n=3\cdot 2^a-3,a\geq 4\}+\{4|n=2^a+7,a\geq 3\}\\
&+\{4|n=2^a+3,a\geq 4\},
\end{aligned}
$$

which is congruent to

$$
\begin{aligned}
&\{5|n=2^a-6,a\geq 5\}+\{1|n=2^a-3,a\geq 3\}+\{6|n=2^a-2,a\geq 3\}\\
&+\{6|n=2^a+3,a\geq 3\}+\{4|n=2^a+9,a\geq 3\}\\
&+\{4|n=3\cdot 2^a-6,a\geq 3\}+\{4|n=3\cdot 2^a-3,a\geq 4\}.
\end{aligned}
$$

Next, we collect all 2-parameter sets. These contribute

$$\{4|n=2^a+2^b+1,2^a+2^b-2,a>b\geq 2\}+\{2|n=2^a+2^b,a>b\geq 2\}$$

$$\begin{aligned}
&+\{4|n=2^a+2^b+4,2^a+2^b+1,a>b\ge 2\}+\{2|n=2^a+2^b+3,a>b\ge 2\}\\
&+\{4|n=2^a+2^b,2^a+2^b+3,2^a+2^b-6,2^a+2^b-3,a>b\ge 2\}\\
&+\{2|n=2^a+2^b,a>b\}+\{4|n=2^a+2^b-3,a>b\ge 2\}\\
&+\{2|n=2^a+2^b-6,a>b\ge 3\}-\{2|n=2^a+2^b+3,a>b\}\\
&+\{4|n=2^a+2^b,a>b\ge 2\}-\{2|n=2^a+2^b-3,a>b\ge 3\}\\
&+\{4|n=2^a+2^b+1,a>b\ge 2\}+\{4|n=2^a+2^b-2,a>b\ge 2\}\\
&+\{4|n=2^a+2^b+1,a>b\ge 2\}+\{4|n=2^a+2^b+4,a>b\ge 2,a\ge 4\},
\end{aligned}$$

which is congruent to

$$\begin{aligned}
&\{6|n=2^a+2^b-6,a>b\ge 2\}+\{6|n=2^a+2^b-3,a>b\ge 3\}\\
&\quad+\{2|n=2^a-2,a\ge 4\}+\{4|n=2^a+2^b+3,a>b\ge 2\}.
\end{aligned}$$

Finally, the contribution coming from 3-parameter sets is

$$\begin{aligned}
&\{4|n=2^a+2^b+2^c,a>b>c\ge 2\}+\{4|n=2^a+2^b+2^c+3,a>b>c\ge 2\}\\
&+\{4|n=2^a+2^b+2^c-6,a>b>c\ge 2\}\\
&+\{4|n=2^a+2^b+2^c+3,a>b>c\ge 2\}\\
&+\{4|n=2^a+2^b+2^c-3,a>b>c\ge 2\}+4\#\{n=2^a+2^b+2^c,a>b>c\ge 2\},
\end{aligned}$$

which is congruent to

$$\{4|n=2^a+2^b+2^c-6,a>b>c\ge 2\}+\{4|n=2^a+2^b+2^c-3,a>b>c\ge 2\}.$$

Now Theorem 2 can be checked by direct inspection. Obviously, the surviving 2- and 3-parameter sets are the ones described in the theorem. The term $\{6|n=2^a-2,a\ge 3\}$ cancels against the corresponding term stemming from the 2-parameter sets, the term $\{4|n=2^a+9,a\ge 3\}$ is the reason for the exceptional condition $b\ge 4$ in the 3-parameter family $n=2^a+2^b+2^c-3$, and the terms $\{4|n=3\cdot 2^a-6,a\ge 3\}$ and $\{4|n=3\cdot 2^a-3,a\ge 4\}$ cause the difference between $s_n\equiv 2$ and $s_n\equiv 6$.

5. Proof of Theorem 3

From [4, Theorem B] we infer that $f_\lambda\equiv 0\,(3)$ for λ even, and that, for λ odd,

$$f_\lambda\equiv(-1)^{(\lambda+1)/2}\,\lambda^{-1}\binom{\lambda}{\frac{\lambda-1}{2}}\pmod 3,$$

implying $f_\lambda\equiv 0\,(3)$ for odd λ, unless

$$\mathfrak{s}_3\Big(\frac{\lambda-1}{2}\Big)+\mathfrak{s}_3\Big(\frac{\lambda+1}{2}\Big)-\mathfrak{s}_3(\lambda-1)=1.$$

This equation can be rewritten as

$$2\mathfrak{s}_3\Big(\frac{\lambda-1}{2}\Big)-\mathfrak{s}_3(\lambda-1)+\underbrace{\Big(\mathfrak{s}_3\Big(\frac{\lambda+1}{2}\Big)-\mathfrak{s}_3\Big(\frac{\lambda-1}{2}\Big)\Big)}_{=-2\nu_3((\lambda+1)/2)+1}=1.$$

Denote by $c(\lambda)$ the number of carries in the multiplication $2\cdot\frac{\lambda-1}{2}$. Then

$$2\mathfrak{s}_3((\lambda-1)/2)-\mathfrak{s}_3(\lambda-1)=2c(\lambda),$$

and we obtain the equation

$$c(\lambda) = \nu_3\Big(\frac{\lambda+1}{2}\Big).$$

The number $c(\lambda)$ is at least the number of digits 2 occurring in the 3-adic expansion of $(\lambda-1)/2$, which in turn is greater or equal to the number of consecutive digits 2 at the right-hand end of $(\lambda-1)/2$; the last quantity being equal to $\nu_3(\frac{\lambda+1}{2})$. Hence, all carries correspond to digits 2, and all digits 2 form a single block at the right-hand end. Let $a \geq 0$ be the length of this block. If $a \geq 1$, then there is a carry at the a-th position from the right, hence, the digit in position $a+1$ must be zero, since otherwise an extra carry would occur at this position. Thus, the sequence of digits of $(\lambda-1)/2$ either does not contain a 2, or consists of an initial section not containing 2, followed by a 0 and a block of $a \geq 1$ digits 2. Consequently, $f_\lambda \not\equiv 0\,(3)$ if and only if $\lambda - 1$ is either a Cantor number, or is of the form $3^{a+1}n + 2\cdot 3^a - 2$ with $a \geq 1$ and n a Cantor number. Define a function $f(n)$ via

$$f(n) := \prod_{\substack{\nu\leq n\\ 3\nmid\nu}} \nu.$$

Then

$$f(n) \equiv \begin{cases} 1, & n \equiv 0,1,5\,(6)\\ 2, & n\equiv 2,3,4\,(6)\end{cases} \pmod 3$$

and

$$\frac{n!}{3^{\nu_3(n!)}} = \prod_k f\big(\lfloor n/3^k\rfloor\big).$$

Note that, if $n \equiv m\,(3)$ and $n \not\equiv m\,(2)$, we have $f(n) \equiv -f(m)\,(3)$, and that, for n even, we have $f(n) \equiv n^2+1\,(3)$. Hence, writing $n = \sum_i a_i 3^i$ with $a_i \in \{0,1,2\}$, we find that modulo 3

$$\begin{aligned} f\big(\lfloor n/3^k\rfloor\big) &\equiv (-1)^{\lfloor n/3^k\rfloor}(a_k^2+1)\\ &\equiv (-1)^{\sum_{i\geq k} a_i}(a_k^2+1), \end{aligned}$$

and therefore

$$\frac{n!}{3^{\nu_3(n!)}} \equiv (-1)^{\sum_k (k+1)a_k + |\{k:a_k\neq 0\}|}.$$

From this, we deduce that

$$\binom{\lambda}{\frac{\lambda-1}{2}} 3^{-\nu_3(\lambda)} \equiv (-1)^s \pmod 3,$$

where

$$\begin{aligned} s = \sum_k (k+1)\big(a_k(\lambda) + a_k((\lambda-1)/2) + a_k((\lambda+1)/2)\big)\\ + |\{k : a_k(\lambda)\neq 0\}| + |\{k : a_k((\lambda-1)/2)\neq 0\}| + |\{k : a_k((\lambda+1)/2)\neq 0\}|. \end{aligned}$$

Now assume that $\lambda-1$ is of the form $3^{a+1}n+2\cdot 3^a-2$ with $a\geq 1$ and n a Cantor number. For $k\geq a+1$, we have either $a_k(\lambda)=a_k((\lambda-1)/2)=a_k((\lambda+1)/2)=0$, or $a_k(\lambda)=2$ and $a_k((\lambda-1)/2)=a_k((\lambda+1)/2)=1$; hence, in both cases,

$$a_k(\lambda)+a_k((\lambda-1)/2)+a_k((\lambda+1)/2)\equiv 0 \pmod 2.$$

We find that the total contribution to s of terms with $k\geq a+1$ is congruent modulo 2 to $|\{k : a_k(\lambda)\neq 0, k\geq a+1\}|$. We have $a_a(\lambda)=1$, $a_a((\lambda-1)/2)=0$, and $a_a((\lambda-1)/2)=1$; thus, there is no contribution arising from position a. Furthermore, $a_0(\lambda)=a_0((\lambda-1)/2)=2$, $a_0(((\lambda+1)/2)=0$, and, for $1\leq i\leq a-1$, $a_0(\lambda)=a_0((\lambda-1)/2)=a_0(((\lambda+1)/2)=2$; hence, the total contribution of values $k\leq a$ is congruent to $a-1$ modulo 2. Thus,

$$\binom{\lambda}{\frac{\lambda-1}{2}}3^{-\nu_3(\lambda)}\equiv(-1)^{a-1+\mathfrak{s}_3(n)/2} \pmod 3,$$

and, since $\lambda\equiv 2\,(3)$, we find that

$$f_\lambda\equiv(-1)^{(\lambda-1)/2+a-1+\mathfrak{s}_3(n)/2} \pmod 3.$$

Moreover, from the discussion of the digits of $(\lambda-1)/2$ above, we deduce that $\mathfrak{s}_3((\lambda-1)/2)=\mathfrak{s}_2(n)/2+2a$, and from the congruence $n\equiv\mathfrak{s}_3(n)\,(2)$ we conclude that $(\lambda-1)/2+\mathfrak{s}_3(n)/2$ is even. Hence, our claim follows in this case.

Now assume that $\lambda-1$ is a Cantor number. From the discussion of the case $k\geq a+1$ given above, we find that $s\equiv\mathfrak{s}_3(\lambda-1)/2\,(2)$ and $(\lambda+1)/2\equiv 1+\mathfrak{s}_3(\lambda-1)/2\,(2)$, and therefore

$$f_\lambda\equiv-\lambda^{-1}3^{\nu_3(\lambda)} \pmod 3.$$

Since $\lambda-1$ is a Cantor number, we have $\lambda\equiv 3^{\nu_3(\lambda)}\,(3^{\nu_3(\lambda)+1})$, which implies $f_\lambda\equiv 2\,(3)$, proving our claim in this case as well. Hence, the first part of Theorem 3 is proven.

Denote by $C(x)$ the number of odd Cantor numbers $n\leq x$. Then $C(x)\asymp x^{\log 2/\log 3}$. Hence, we obtain the estimate

$$C(x)\leq N(x)\leq C(x)+\sum_{k\geq 1}C(x/3^{k+1}).$$

Thus, we have $N(x)\asymp C(x)$, and the second claim follows as well.

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School of Mathematical Sciences, Queen Mary & Westfield College, University of London, Mile End Road, E1 4NS London, UK
T.W.Muller@qmul.ac.uk

Universität Freiburg, Mathematisches Institut, Eckerstr. 1, 79104 Freiburg, Germany
jcp@math.uni-freiburg.de

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