The equation $\omega(n) = \omega(n+1)$

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Abstract

We prove that there are infinitely many integers n such that n and $n+1$ have the same number of distinct prime divisors.

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In this note we prove the following theorem:

Theorem 1 *There are infinitely many n such that n and $n+1$ have the same number of distinct prime factors.*

Our argument will be similar to the one given by D. R. Heath-Brown[1] for consecutive integers with the same number of divisors. The method given there can be applied to a variety of similar problems involving multiplicative functions f such that $f(p^e)$ is a nonconstant function of e for $e \geq 1$, but fails here, since $2^{\omega(p^e)}$ is clearly independent of e. Thus different from [1], where arbitrary large special sets were constructed in a systematic way, we have to construct our special set numerically. Since we weren't able to find one with more than 5 elements, we have to use rather sharp sieve estimates, provided by the following estimate, which is a special case of a theorem of D. R. Heath-Brown[2]:

Theorem 2 *Let $c_1, \dots c_5$ be natural numbers. Then there are infinitely many natural numbers n, such that*

$$\sum_{i=1}^{5} 2^{\omega(c_i n+1)} \leq 57$$

Now we prove Theorem 1. Assume that there are 5 integers $a_1, \dots, a_5$ with the properties that for any $1 \leq i < j \leq 5$ we have $(a_i, a_j) = |a_i - a_j|$, and $\omega\left(\frac{a_i}{|a_i-a_j|}\right) = \omega\left(\frac{a_j}{|a_i-a_j|}\right)$. Define $A = a_1 \cdots a_5$, and set $c_i = a_i A$. Let n be an integer as described by Theorem 2. Then for some pair $i \neq j$ we have $\omega(a_i An+1) = \omega(a_j An+1)$, for otherwise the left hand side sum would be bounded below by $2^1 + 2^2 + \ldots + 2^5 = 62$. Then consider the integers $\frac{a_j(a_i An+1)}{|a_i-a_j|}$ and $\frac{a_i(a_j An+1)}{|a_i-a_j|}$. They are obviously consecutive, so it suffices to show that they have the same number of prime divisors. Since $a_i|A$, a_i and a_jAn+1 are coprime and we get

$$\begin{aligned}\omega\left(\frac{a_j(a_iAn+1)}{|a_i-a_j|}\right) &= \omega\left(\frac{a_i}{|a_i-a_j|}\right)+\omega(a_iAn+1)\\ &= \omega\left(\frac{a_j}{|a_i-a_j|}\right)+\omega(a_jAn+1)\\ &= \omega\left(\frac{a_i(a_jAn+1)}{|a_i-a_j|}\right)\end{aligned}$$

Hence for any n given by Theorem 2 we obtain one pair of consecutive integers with the same number of distinct prime divisors, and any pair can come only from finitely many values of n, hence there are infinitely many such pairs.

It remains to find $a_1, \dots, a_5$ with the described properties. The problem is that the ten equations to be checked are not independent. To illustrate this, we show that in such a set of a_i no three can be consecutive integers. Suppose to the contrary that $a+3 = a_2+1 = a_3+2$. Then $(a_3, a_1) = a_3 - a_1 = 2$ and $2|a_1$, and so taking $N = a_1/2$ gives the three equations $\omega(2N) = \omega(2N+1)$, $\omega(2N+1) = \omega(2N+2)$ and $\omega(N) = \omega(N+1)$. The first two equations imply $\omega(2N) = \omega(2N+2)$. If N was even, $\omega(2N) = \omega(N)$ whereas $\omega(2N+2) = \omega(N+1)+1$, which gives a contradiction, and in the same way the case N odd can be ruled out.

Such obstructions can be avoided, if one chooses the a_i in such a way, that the differences $|a_i - a_j|$ are divisible by many different prime factors, for by choosing additional congruence restrictions one can avoid situations as above. On the other hand, the differences should be fairly small, for otherwise the a_i would become very large, which would increase the computational effort to check a given quintuple.

After some experimentation, the following set of quintuples seemed promising: Define $b_1 = 8, b_2 = 9, b_3 = 12, b_4 = 34, b_5 = 576$, $N = 2^4 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 47^2 \cdot 71^2 \cdot 271$, $k = 110245379356152833616$ and consider the sequence of quintuples $(l \cdot N + k + b_1, \dots, l \cdot N + k + b_5)$. In fact, for $l = 1202$ this gave the following quintuple

$$\begin{aligned}a_1 &= 135987650281178292389624\\ &= 2^3 \cdot 13 \cdot 29 \cdot 71^2 \cdot 431 \cdot 733 \cdot 28311976573\\ a_2 &= 135987650281178292389625\end{aligned}$$

$$
\begin{aligned}
&= \quad 3^5 \cdot 5^3 \cdot 7 \cdot 1481 \cdot 3109 \cdot 80737 \cdot 1720429 \\
a_3 &= 1359876502811782923896 28 \\
&= \quad 2^2 \cdot 3 \cdot 11 \cdot 31^2 \cdot 47^2 \cdot 53 \cdot 6899 \cdot 1327224593 \\
a_4 &= 135987650281178292389650 \\
&= \quad 2 \cdot 5^2 \cdot 11^2 \cdot 13 \cdot 19 \cdot 271 \cdot 1145107 \cdot 293245787 \\
a_5 &= 135987650281178292390192 \\
&= \quad 2^4 \cdot 3^4 \cdot 7^2 \cdot 47 \cdot 71 \cdot 271 \cdot 2367951977749
\end{aligned}
$$

Together with the factorization of the differences, i.e.

$$
\begin{aligned}
a_2 - a_1 &= 1 & a_4 - a_2 &= 25 = 5^2 \\
a_3 - a_1 &= 4 = 2^2 & a_5 - a_2 &= 567 = 3^4 \cdot 7 \\
a_4 - a_1 &= 26 = 2 \cdot 13 & a_4 - a_3 &= 22 = 2 \cdot 11 \\
a_5 - a_1 &= 568 = 2^3 \cdot 71 & a_5 - a_3 &= 564 = 2^2 \cdot 3 \cdot 47 \\
a_3 - a_2 &= 3 & a_5 - a_4 &= 542 = 2 \cdot 271
\end{aligned}
$$

it is easy to check that these values of a_i satisfy all necessary properties, and therefore Theorem 1 is proven.

The computations were performed using Mathematica 4.1.

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References

[1] D. R. Heath-Brown, *Almost-prime k-tuples*, Mathematika 44, 245–266, (1997)

[2] D. R. Heath-Brown, *The divisor function at consecutive integers*, Mathematika 31, 141–149, (1984)

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