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A p-group with positive rank gradient

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Abstract. We construct, for $d \geq 2$ and $\epsilon > 0$, a d-generated p-group Γ which, in an asymptotic sense, behaves almost like a d-generated free pro-p-group. We show that a subgroup of index p^n needs $(d - \epsilon)p^n$ generators, and that the subgroup growth of Γ satisfies $s_{p^n}(\Gamma) > s_{p^n}(F_d^p)^{1-\epsilon}$, where F_d^p is the d-generated free pro-p-group. To do this we introduce a new invariant for finitely-generated groups and study some of its basic properties.

1 Introduction and results

Burnside asked in 1902 whether a finitely-generated torsion group is necessarily finite. This question remained open until 1964, when Golod [5] constructed an infinite, finitely-generated p-group. Since then there have been several other constructions of finitely-generated infinite p-groups, such as Gupta and Sidki's construction [6] using automorphisms of the p-adic tree. After Burnside's question was settled, most subsequent investigations considered the question of whether finitely-generated periodic could be replaced by a stronger restriction which still allows for infinite groups. Much less work has been done on the question of how large a finitely generated torsion group could be. Here we show that such a group can, in some sense, be almost as large as a free group with the same number of generators. Denote by $d(G)$ the number of generators of G. Let Γ be a finitely-generated residually-finite infinite group, $\mathcal{N} = (N_i)_{i=1}^{\infty}$ be a sequence of finite-index normal subgroups of Γ with $\bigcap N_i = \{1\}$. Define the rank gradient $\mathrm{rg}(\Gamma, \mathcal{N})$ of Γ along $\mathcal{N}$ as $\liminf \frac{d(N_i)}{(\Gamma:N_i)}$, and the rank gradient of Γ as the infimum of $\mathrm{rg}(\Gamma, \mathcal{N})$ taken over all sequences N_i of finite-index normal subgroups with trivial intersection. Obviously, $\mathrm{rg}(\Gamma) \leq d(\Gamma) - 1$. This invariant was introduced by Lackenby [8] in connection with the virtual Haken conjecture for 3-manifolds; Abért and Nikolov [2] showed that it is also related to topological dynamics. In relation to the virtual positive b_1-conjecture, Nikolov asked whether there exists a group of positive rank gradient such that no finite-index subgroup maps onto an infinite cyclic group. Here we show that such a group indeed exists, in fact, we prove the following.

Theorem 1. *Let $d \geq 2$ be an integer, p a prime, and $\epsilon > 0$. Then there exists a d-generated p-group Γ with* $\mathrm{rg}(\Gamma) \geq d - 1 - \epsilon$.

The existence of finitely-generated p-groups with positive rank gradient was independently shown by Osin [11]. Using results by Lackenby [7] and Abert, Jaikin-Zapirai and Nikolov [1], we obtain the following.

Corollary 1. *There exists an infinite non-amenable torsion group.*

This result was recently proven by Ershov [4] (see also [11, Corollary 1.3]), however, the proof given here is more elementary.

For the proof we define another invariant, the p-deficiency of a group Γ. Let Γ be a group with a presentation $\langle X|R\rangle$, where we assume X to be finite. We can view an element $r \in R$ as an element in the free group $F(X)$ over X, and define the p-order $\nu_p(r)$ of r to be the largest integer k for which there is some $s \in F(X)$ with $r = s^{p^k}$. Define the p-deficiency of $\langle X|R\rangle$ as $|X| - 1 - \sum_{r\in R} p^{-\nu_p(r)}$, and the p-deficiency $\mathrm{def}_p(\Gamma)$ of Γ as the supremum taken over all presentations of Γ with finite generating set. Our main technical result is the following.

Theorem 2. *Let Γ be an infinite group, $\Delta \triangleleft \Gamma$ a normal subgroup of index p. Then* $\mathrm{def}_p(\Delta) \geq p \cdot \mathrm{def}_p(\Gamma)$.

From this we deduce the following. For a group Γ denote by $s_n(\Gamma)$ the number of subgroups of index n, and by $s_n^{\triangleleft\triangleleft}(\Gamma)$ the number of subnormal subgroups of index n.

Proposition 1. (1) *If Γ is a p-group which is residually finite, then*

$$\mathrm{def}_p(\Gamma) \leq \mathrm{rg}(\Gamma).$$

(2) *For every $\epsilon > 0$ we have* $s_{p^n}^{\triangleleft\triangleleft}(\Gamma) \gg p^{(\mathrm{def}_p(\Gamma)-\epsilon)\frac{p^{n+1}}{p-1}}$.

(3) *For every $d \geq 2$, p a prime and $\epsilon > 0$ there exists a d-generated p-group Γ with* $\mathrm{def}_p(\Gamma) \geq d - 1 - \epsilon$.

(4) *For every $\epsilon > 0$ there exists a d-generated torsion pro-p-group Γ with*

$$s_{p^n}(\Gamma) \gg s_{p^n}(\widehat{F_d})^{1-\epsilon},$$

where $\widehat{F_d}$ is the d-generated free pro-p-group.

We can define the p-deficiency for pro-finite groups as we did for discrete groups, we just replace the rank by the topological rank, that is, the minimal number of elements generating a dense subgroup. In the case of pro-p-groups this allows us to construct an Euler-characteristic for finitely-generated pro-p-groups as follows. We define

$$\chi_p(\Gamma) = -\sup_{\Delta < \Gamma,\, \Delta \text{ open}} \frac{\operatorname{def}_p(\Delta)}{(\Gamma : \Delta)}.$$

Note that the sign is somewhat arbitrary, we chose it to be consistent with the Euler-characteristic of a virtually free group. For this invariant we prove the following.

Proposition 2. (1) *If Γ is a pro-p-group, then $-\operatorname{rg}(\Gamma) \leq \chi_p(\Gamma) \leq -\operatorname{def}_p(\Gamma)$.*

(2) *If $\chi_p(\Gamma) < 0$, and Δ is an open subgroup of Γ, then $\chi_p(\Delta) = (\Gamma : \Delta)\chi_p(\Gamma)$.*

(3) *Let Γ be a virtually-free group, which contains a normal free subgroup of p-power index, $\widehat{\Gamma}$ be the pro-p-completion of Γ. Then $\chi_p(\widehat{\Gamma}) = \chi(\Gamma)$.*

(4) *Let Γ be a Fuchsian group, which contains a normal surface group as a subgroup of p-power index, $\widehat{\Gamma}$ be the pro-p-completion of Γ. Then we have $\chi_p(\widehat{\Gamma}) = -\mu(\Gamma)$, where μ is the hyperbolic volume of a group.*

In spite of Theorem 1 above, positive p-deficiency implies that a group behaves roughly like a large group. For example, the following is an immediate consequence of Lackenby's characterization of finitely presented large groups (cf. [9]).

Proposition 3. *A finitely presented group Γ with positive p-deficiency contains a finite-index subgroup Δ which projects surjectively onto a non-abelian free group.*

2 Proof of Theorem 2

Let Γ be a group with a presentation $\langle X | R \rangle$, where $|X| = d$ is finite. Let F be the free group over X. For a subgroup $H \leq F$ and an element $g \in F$, we define

$$g^H = \{g^h : h \in H\}$$

to be the H-conjugacy class of g. Subgroups of free groups are free, and since the p-order of an element is only defined with respect to one fixed free group containing it, we define $\nu_{p,H}(g)$ for $g \in H$ to be the largest k for which there exists some $h \in H$ with $h^{p^k} = g$.

Lemma 1. *Suppose that $N \lhd F$ is a normal subgroup of index p. Then we have $\nu_{p,N}(g) \geq \nu_{p,F}(g) - 1$ for all $g \in N$.*

Proof. If $h^{p^k} = g$, then $h^p \in N$ and so $\nu_{p,N}(g) \geq k - 1$. □

Lemma 2. *Let $N \lhd F$ be of index p and let $g \in N$. Then we have either $g^F = g^N$ or*

$$g^F = \bigcup_{i=1}^{p} g_i^N.$$

Moreover, in the latter case we have $\nu_{p,N}(g) = \nu_{p,F}(g)$.

Proof. The group F acts on g^F transitively by conjugation, so the normal subgroup N has either 1 or p orbits. The latter is equivalent to saying that the stabilizer of g in H equals the stabilizer of g in F, that is, that the centralizer $C_F(g)$ satisfies $C_F(g) \leq N$. Assume that this is the case. If $h^{p^k} = g$, then h and g commute, hence $h \in N$. So $\nu_{p,N}(g) = \nu_{p,F}(g)$. □

We now prove Theorem 2. Let $\langle X|R\rangle$ be a presentation for Γ, $\Delta \lhd \Gamma$ a normal subgroup of index p, and let F be the free group over X. Let $\phi : F \to \Gamma$ be the homomorphism defined by the presentation, K the kernel of ϕ, and $N = \phi^{-1}(\Delta)$ the preimage of Δ. Then $N \lhd F$ is a normal subgroup of index p and, by the Nielsen–Schreier Theorem, N is generated by $n = (d-1)p + 1$ elements; we fix a generating set $Y = y_1, \dots, y_n$.

We construct a presentation $\langle Y|S\rangle$ for Δ using this generating set and K. Now K is generated by the conjugacy classes r^F, $r \in R$, as a subgroup. Let us use Lemma 2. If for some $r \in R$ we have $r^F = r^N$, then we add r to S, expressed as a word over Y. In this case, by Lemma 1, we have $\nu_{p,N}(r) \geq \nu_{p,F}(r) - 1$. Otherwise, r^F is the disjoint union of p conjugacy classes under N; let us add one element from each conjugacy class to the presentation as a relation s. In this case, by Lemma 2, we have $\nu_{p,N}(r) = \nu_{p,F}(r)$. We do so for every $r \in R$, and collect the resulting relations into the set S. Then $\langle Y|S\rangle$ will be a presentation for Δ. Using these estimates we get

$$\begin{aligned}
\operatorname{def}_p(\langle Y|S\rangle) &\geq n - 1 - \sum_{s\in S} p^{-\nu_{p,N}(s)}\\
&\geq p(d-1) - p\sum_{r\in R} p^{-\nu_{p,F}(r)}\\
&= p \cdot \operatorname{def}_p(\langle X|R\rangle).
\end{aligned}$$

Let $\epsilon > 0$ be given. Then there exists a presentation $\Gamma = \langle X|R\rangle$ of Γ such that $\operatorname{def}_p(\Gamma) \leq \operatorname{def}_p(\langle X|R\rangle) - \epsilon$. From the computation above we see that there exists a presentation $\langle Y|S\rangle$ of Δ with $\operatorname{def}_p(\langle Y|S\rangle) \geq p \cdot \operatorname{def}_p(\langle X|R\rangle)$, hence we obtain $\operatorname{def}_p(\Delta) \geq p \cdot (\operatorname{def}_p(\Gamma) - \epsilon)$. Since ϵ is arbitrary, Theorem 2 follows.

Note that elements of S may be redundant, or there could be more economic presentations of Δ, hence, in general we do not have equality.

3 Proofs of the other statements

We begin by proving Proposition 1 (1).

Let Γ be a p-group. If $\mathrm{def}_p(\Gamma) \leq 0$, there is nothing to show. Hence assume that $\mathrm{def}_p(\Gamma) > 0$. Note first that Γ contains a normal subgroup Δ of index n, since in C_p all relators r with $\nu_p(r) < 1$ are trivial. Since $\mathrm{def}_p(\Delta) > 0$, we conclude that Γ contains subgroups of arbitrary large finite-index.

Let Δ be a normal subgroup of index p^k. Then there exists a chain

$$\Gamma = \Delta_0 > \Delta_1 > \cdots > \Delta_k = \Delta$$

such that $(\Delta_i : \Delta_{i+1}) = p$. From Theorem 2 we obtain $\mathrm{def}_p(\Delta) \geq p^k \mathrm{def}_p(\Gamma)$. Hence, Δ has a presentation $\langle X | R \rangle$ such that

$$|X| - 1 - \sum_{r \in R} p^{-\nu_p(r)} \geq p^k \mathrm{def}_p(\Gamma) - \epsilon.$$

Let $R_1 \subseteq R$ be the set of relations that are not p-th powers. Every $r \in R_1$ contributes 1 to the left hand side sum, hence, $|X| - 1 - |R_1| \geq p^k \mathrm{def}_p(\Gamma) - \epsilon$. We now bound $|\mathrm{Hom}(\Delta, C_p)|$. This number equals the number of solutions of the system $\{r = 1 \mid r \in R\}$, where the variables are from C_p, and the equations are to be interpreted as equations in C_p. Since every p-th power is trivial in C_p, the equations from $R \setminus R_1$ are trivially satisfied, and we see that the original system is equivalent to the system $\{r = 1 \mid r \in R_1\}$. This system can be viewed as a system of $|R_1|$ linear equations in $|X|$ variables over the field with p elements, hence there are at least

$$p^{|X| - |R_1|} > p^{p^k \mathrm{def}_p(\Gamma)}$$

solutions. Then the p-Frattini quotient of Δ has cardinality at least $p^{p^k \mathrm{def}_p(\Gamma)}$, which implies that Δ has an elementary abelian p-group of rank at least $p^k \mathrm{def}_p(\Gamma)$ as quotient. But then Δ itself has rank at least $p^k \mathrm{def}_p(\Gamma)$. Hence, for every normal subgroup Δ of p-power index we have that Δ has rank at least $(\Gamma : \Delta)\mathrm{def}_p(\Gamma)$, that is, $\mathrm{rg}(\Gamma) \geq \mathrm{def}_p(\Gamma)$.

Now we prove part (2). We have $s_{p^n}^{\lhd\lhd}(\Gamma) = s_{p^n}(\widehat{\Gamma})$, where $\widehat{\Gamma}$ is the pro-p-completion of Γ. For a pro-p-group Γ let $\mathrm{sc}_n(\Gamma)$ to be the number of subgroup chains $\Gamma = \Delta_0 > \Delta_1 > \cdots > \Delta_n$, where $(\Delta_i : \Delta_{i+1}) = p$. The quantities $\mathrm{sc}_n(\Gamma)$ and $s_{p^n}(\Gamma)$ are linked via the following (cf. [12]).

Lemma 3. *For a pro-p-group Γ we have*

$$\mathrm{sc}_n(\Gamma) \geq s_{p^n}(\Gamma) \gg p^{-n^2} \mathrm{sc}_n(\Gamma).$$

As in the proof of part (1) we find that the p-Frattini quotient of a subgroup of index p^k in Γ^p has rank at least $p^k \mathrm{def}_p(\Gamma)$, that is, a subgroup of index p^k has at least $\frac{p^{p^k}\operatorname{def}_p(\Gamma)-1}{p-1}$ subgroups of index p. Hence, the number of subgroup chains of length n is at least

$$\begin{aligned}
\operatorname{sc}_n(\Gamma^p) &\geq \frac{p^{p^{k_0}}\operatorname{def}_p(\Gamma)-1}{p-1}\frac{p^{p^{k_0+1}}\operatorname{def}_p(\Gamma)-1}{p-1}\cdots\frac{p^{p^{n-1}}\operatorname{def}_p(\Gamma)-1}{p-1}\\
&\gg p^{\frac{p^n-1}{(p-1)}\operatorname{def}_p(\Gamma)}(p-1)^{-n}\\
&\geq p^{\frac{p^n-1}{(p-1)}(\operatorname{def}_p(\Gamma)-\epsilon)},
\end{aligned}$$

where k_0 is chosen sufficiently large to ensure that $p^{p^{k_0}}\operatorname{def}_p(\Gamma)-1$ is positive. Our claim now follows by combining these estimates.

We construct a d-generated p-group with p-deficiency close to $d-1$ as follows. Number the words in F_d in some way as $w_1, \ldots$, and choose some integer k. Then define the group

$$\Gamma = \langle x_1, \ldots, x_d | w_1^{p^k}, w_2^{p^{k+1}}, w_3^{p^{k+2}}, \ldots\rangle.$$

Every element of Γ is a word in $x_1, \ldots, x_d$, hence every element has order a power of p. The p-deficiency of this presentation is

$$d-1-\sum_{\nu=k}^{\infty} p^{\nu} = d-1-\frac{p^{1-k}}{p-1} \geq d-1-2p^{-k},$$

hence, for k large enough, we have $\operatorname{def}_p(\Gamma) > d-1-\epsilon$.

Finally, the fourth statement follows immediately from the previous two.

Theorem 1 does not follow immediately from Proposition 1 (1) and (3), since the group constructed in (3) is not necessarily residually-finite. In fact, factoring out the residual might change the p-deficiency of a group in an unforeseen way. Let Γ be a p-group with residual N, and Δ a finite-index subgroup. Then $\operatorname{def}_p(\Delta)$ is a lower bound for $\dim \Delta/([\Delta,\Delta]\Delta^p)$, and since $[\Delta,\Delta]\Delta^p$ contains N, it follows that $\dim \Delta/([\Delta,\Delta]\Delta^p)$ is a lower bound for $d(G)$. Hence, when constructing a group as in (3), its maximal residual finite quotient will satisfy the requirements of Theorem 1.

Corollary 1 follows immediately from Theorem 1 and the fact that a group of positive rank gradient cannot be amenable. This was proven by Lackenby [7] for finitely-presented groups, and by Abert, Jaikin-Zapirai and Nikolov [1] in the general case.

Lackenby [9] showed that a finitely-presented group Γ which has positive rank gradient along a series $N_1 > N_2 > \cdots$ of normal subgroups, such that N_i/N_{i+1} is abelian, is large. From this, Proposition 3 follows immediately on observing Proposition 1 (1).

4 Towards an Euler characteristic

Here we prove Proposition 2. The inequality $\chi_p(\Gamma) \leq -\mathrm{def}_p(\Gamma)$ follows from the fact that the supremum of a function taken over all finite-index subgroups is at least the value at the group itself. For the inequality $-\mathrm{rg}(\Gamma) \leq \chi_p(\Gamma)$ note first that, since Γ is a pro-p-group, we have that for all open subgroups Δ of Γ there exists a chain $\Gamma = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_k = \Delta$ with $N_i/N_{i+1} \cong C_p$. Hence we have that $\mathrm{def}_p(\Delta) \geq (\Gamma : \Delta)\mathrm{def}_p(\Gamma)$ for all open subgroups Δ of Γ.

Let (N_i) be a descending sequence of normal open subgroups with trivial intersection such that

$$\lim_{i\to\infty} \frac{d(N_i)}{(\Gamma : N_i)} \leq \mathrm{rg}(\Gamma) + \epsilon,$$

and choose an index i such that $d(N_i) \leq \mathrm{rg}(\Gamma) + 2\epsilon$. Let Δ be an open subgroup of Γ with $\mathrm{def}_p(\Delta) > (-\chi_p(\Gamma) - \epsilon)(\Gamma : \Delta)$. Then

$$\mathrm{def}_p(\Delta \cap N_i) \geq (\Delta : \Delta \cap N_i)\mathrm{def}_p(\Delta),$$

hence we may assume that Δ is a subgroup of N_i. On the other hand we have the estimate $d(\Delta) \leq (N_i : \Delta)d(N_i)$, hence

$$\begin{aligned}
-\chi_p(\Gamma) &\leq \frac{\mathrm{def}_p(\Delta)}{(\Gamma : \Delta)} + \epsilon \\
&\leq \frac{d(\Delta)}{(\Gamma : \Delta)} + \epsilon \\
&\leq \frac{d(N_i)(N_i : \Delta)}{(\Gamma : \Delta)} + \epsilon \\
&\leq \frac{(\mathrm{rg}(\Gamma) + 2\epsilon)(\Gamma : N_i)(N_i : \Delta)}{(\Gamma : \Delta)} + \epsilon \\
&= \mathrm{rg}(\Gamma) + 3\epsilon,
\end{aligned}$$

and our claim follows.

For the multiplicativity, note that for $\Gamma > \Delta$ we have $\chi_p(\Gamma) \geq (\Gamma : \Delta)\chi_p(\Delta)$, since the set over which the supremum is taken to compute the left-hand side is a superset of the set used for the right-hand side. Let $U < \Gamma$ be a finite-index subgroup. Then we have $\mathrm{def}_p(U \cap \Delta) \geq (U : U \cap \Delta)\mathrm{def}_p(U)$, hence

$$-\chi_p(\Delta) \geq \frac{\mathrm{def}_p(U \cap \Delta)}{(\Delta : U \cap \Delta)} \geq \frac{\mathrm{def}_p(U)(U : U \cap \Delta)}{(\Delta : U \cap \Delta)} = \frac{\mathrm{def}_p(U)(\Gamma : \Delta)}{(\Gamma : U)}.$$

Now, choosing for U a subgroup with $\mathrm{def}_p(U) \geq (\Gamma : U)(-\chi_p(\Gamma) - \epsilon)$, we obtain $-\chi_p(\Delta) \geq (\Gamma : \Delta)(-\chi_p(\Gamma) - \epsilon)$, which implies our claim.

If Γ is a free pro-p-group with d generators, then the obvious presentation yields $\mathrm{def}_p(\Gamma) \geq d-1$. On the other hand the rank gradient of Γ is $d-1$, hence we have $\mathrm{def}_p(\Gamma) = -\chi_p(\Gamma) = d-1$. In the same way we find that if Γ is a d-generated Demuskin-group, we have $\mathrm{def}_p(\Gamma) = -\chi_p(\Gamma) = d-2$. Since the pro-$p$-completion of a virtually-free group that contains a free normal subgroup of p-power index has a normal subgroup of the same index, which is free pro-p, the third statement follows. Similarly, the pro-p-completion of a Fuchsian group which contains only elliptic elements of p-power order is virtually free or virtually Demuskin. This implies our last claim.

We remark that χ_p is not multiplicative with respect to direct products. For example, take $\Gamma = \widehat{F_2 \times F_2}$. The rank gradient of this group is the same as the rank gradient of the discrete group $F_2 \times F_2$ and, taking normal subgroups of the form $N \times N$, one sees that the rank gradient of $F_2 \times F_2$ is 0. On the other hand, $\chi_p(\widehat{F_2}) = 1$, thus, χ_p is not multiplicative on direct products.

5 Problems

The most obvious question is whether our definition of p-deficiency is the right one or not. Right now the definition appears rather ad hoc, and it may be difficult to prove anything about this invariant without a more conceptual definition. Therefore we pose the following.

Problem 1. Give a homological characterization of $\mathrm{def}_p(\Gamma)$.

In our definition of def_p we weighted a relator r according to the largest k such that $r = 0$ holds true in every group of exponent p^k. However, it appears more natural to weight according to the largest k such that $r = 0$ holds true in P_k, the p-Sylow subgroup of the symmetric group of S_{p^k}. However, the naïve approach does not work, since all finite-index subgroups of the group $\Gamma = \langle x, y | [x, y] = 1\rangle$ are isomorphic to Γ, that is, if we want a statement like Theorem 2 to hold, we have to assign the weight 1 to $[x, y]$, although applying a commutator pushes elements down the upper central series. Hence, we are looking for the correct weights.

Problem 2. Find a function $w : F_r \to [0, 1]$ such that $w(x^p) \leq \frac{w(x)}{p}$, the w-deficiency defined via

$$\mathrm{def}_w(\langle X|R\rangle) = |X| - 1 - \sum_{r\in R} w(r)$$

satisfies Theorem 2, and $w(x_i) \to 0$ as $x_i \to 1$ in the pro-p-completion F_r^p of F_r.

It appears difficult to compute the p-deficiency. Giving one example of a presentation one can give a lower bound, however, proving the non-existence of a presentation of a certain form appears difficult, unless one has a matching upper bound e.g. by subgroup-growth. However, we believe that in general positive p-deficiency is a much stronger property than large subgroup growth. In fact, if Γ is a group, and G is a finite p-group with many generators, then the free product $\Gamma * G$ should have smaller p-deficiency than Γ, while most other asymptotic parameters should increase. More specifically, we pose the following.

Problem 3. Let G be an elementary abelian group of order 64 and F_2 a free group with two generators. Determine whether $G * F_2$ has positive p-deficiency or not.

We believe that the Golod–Shafarevich inequality should essentially hold true with deficiency replaced by p-deficiency. More precisely, we pose the following.

Problem 4. Let G be a finite group. Prove that there exists a function $f(n)$, tending to infinity with n, such that $\mathrm{def}_p(G) \leq -f(d(G))$.

It appears that the best way to attack this problem is by solving Problem 1 first. The concept of a p-Euler characteristic might also be generalized.

Problem 5. Find a natural function, which is multiplicative on subgroups, and coincides with the Euler-characteristic for virtually-free groups, the hyperbolic volume for Fuchsian groups, and χ_p for pro-p-groups.

Finally, one should be able to generalize the concept to discrete groups without singling out a special prime p. However, the most obvious way of doing so fails, for if p, q are different prime numbers, then the group

$$\langle x, y | x^{p^n} = y^{p^n} = x^{q^n} = y^{q^n} = 1 \rangle$$

has four relators, each of which only slightly affects the size of the group, as, for example, measured by its subgroup growth. However, the coprimality of the exponents forces the whole group to be trivial. We believe that divisibility issues are the only cause for this kind of collapse. More precisely, we pose the following.

Problem 6. Let $w_1, w_2, \dots$ be an enumeration of all elements in F_2, and let (a_n) be a sequence of positive integers. Then the group

$$\Gamma = \langle x, y | w_1(x, y)^{a_1!}, w_2(x, y)^{a_2!}, \dots \rangle$$

is obviously torsion. Prove that for every ϵ we can choose the sequence (a_n) in such a way that $s_n(\Gamma) \gg (n!)^{1-\epsilon}$.

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Bibliography

[1] M. Abért, A. Jaikin-Zapirain and N. Nikolov, The rank gradient from a combinatorial viewpoint, preprint (2007), `http://arxiv.org/abs/math/0701925`.

[2] M. Abért and N. Nikolov, Rank gradient, cost of groups and the rank versus Heegaard genus problem, preprint (2007), `http://arxiv.org/abs/math/0701361`.

[3] B. Baumslag and S. J. Pride, Groups with two more generators than relators, *J. Lond. Math. Soc. (2)* **17** (1978), 425–426.

[4] M. Ershov, Golod–Shafarevich groups with property (T) and Kac–Moody groups, *Duke Math. J.* **145** (2008), 309–339, 425–426.

[5] E. S. Golod, On nil-algebras and finitely approximable p-groups, *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), 273–276.

[6] N. Gupta and S. Sidki, On the Burnside problem for periodic groups, *Math. Z.* **182** (1983), 385–388.

[7] M. Lackenby, Expanders, rank and graphs of groups, *Israel J. Math.* **146** (2005), 357–370.

[8] M. Lackenby, Heegaard splittings, the virtually Haken conjecture and property (τ), *Invent. Math.* **164** (2006), 317–359.

[9] M. Lackenby, A characterisation of large finitely presented groups, *J. Algebra*, to appear.

[10] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[11] D. Osin, Rank gradient and torsion groups, preprint (2009), `http://arxiv.org/abs/0905.1322`.

[12] J.-C. Schlage-Puchta, Subgroup growth of large pro-p-groups, submitted.

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