Stochastic Calculus in Finance

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Outline

- Motivation and little history
- Binomial model
- Random walk and scaled random walk
- Brownian motion (Wiener process)
- Stochastic analysis
  - stochastic integral,
  - Itô's formula
  - stochastic differential equations
- Black-Scholes-Merton model

Harry M. Markowitz (*1927)

1952 *Portfolio Selection*, The Journal of Finance 7 (1): 77–91.

1952 *The Utility of Wealth*, The Journal of Political Economy (Cowles Foundation Paper 57) LX (2): 151–158.

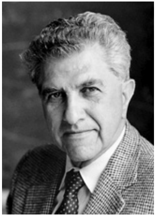
1955 *Portfolio Selection*, Ph.D. thesis at the University of Chicago.

1959 *Efficient Diversification of Investments*, New York: John Wiley & Sons.

Constructed a micro theory of portfolio management for individual wealth holders.

Baruch College, City University of New York,
Rady School of Management, University of California at San Diego

Merton H. Miller (1923-2000)



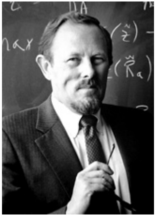
1958 *The Cost of Capital, Corporate Finance and the Theory of Investment*

1972 *The Theory of Finance*, New York: Holt, Rinehart & Winston.

First one with "no arbitrage" argument (no risk-less money machines).

Harward University,
Johns Hopkins University

William F. Sharpe (*1934)



1963 *A Simplified Model for Portfolio Analysis*, Management Science 9 (2): 277–93.

1964 *Capital Asset Prices - A Theory of Market Equilibrium Under Conditions of Risk*, Journal of Finance XIX (3): 425–42.

Binomial method for the valuaiton of options.

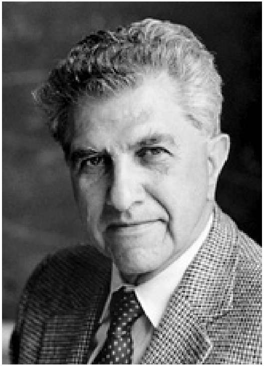
Stanford University,
University of California, Berkeley,
UCLA

Prize in Economic Sciences in Memory of Alfred Nobel

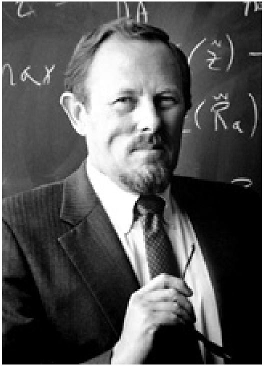
1990 Nobel Prize in Economics:

for their pioneering work in the theory of financial economics

Harry M. Markowitz



Merton H. Miller



William F. Sharp

Robert C. Merton (*1944)

1969 Merton's portfolio problem (consumption vs. investment)

1971 Merton's model for pricing European options (equity = option in firm's asset)

1971 *Theory of rational option pricing*,

1973 ICAPM *International Capital Asset Pricing Model*

First one who uses continuous-time default probabilities to model options on the common stock of a company, i.e. he uses stochastic calculus in finance

Columbia University
California Institute of Technology
Massachusetts Institute of Technology

Myron S. Sholes (*1941)

1973 The pricing options and corporate liabilities,

Together with Fischer Black (1938-1995), the fanous Black-Scholes formula, a fair price for a European call option (i.e. the right to buy one share of a given stock at a specified price and time).

Stanford University

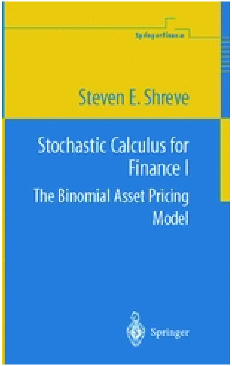
1997 Nobel Prize in Economics:

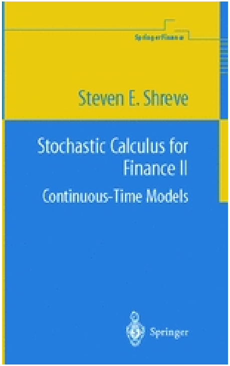
for a new method to determine the value of derivatives

Robert C. Merton

Myron S. Sholes

Stochastic Calculus for Finance I and II





- Steven E. Shreve: *Stochastic Calculus for Finance I, The Binomial Asset Pricing Model*, Springer, New York, 2004.
- Steven E. Shreve: *Stochastic Calculus for Finance II, Continuous-Time Models*, Springer, New York, 2004.

Management Mathematics 1 and 2

KMA/MAM1A: Management Mathematics 1

(4th year, winter term, 2+1, 5 ECTS credicts)

- The Binomial No-Arbitrage Pricing Model
- Probability Theory on Coin Toss Space
- State Prices
- American Derivative Securities
- Random Walk

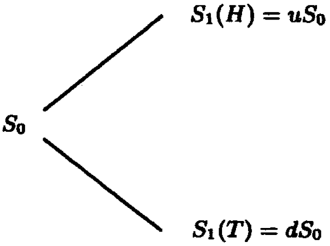
KMA/MAM2A: Management Mathematics 2

(4th year, summer term, 2+1, 5 ECTS credits)

- Stochastic Calculus
- Risk-Neutral Pricing
- Connections with PDEs
- Exotic Options
- American Derivative Securities
- Change of Numéraire

One-Period Binomial Pricing Model

At time t_0: initial stock price is $S_0 > 0$.
We toss a coin: the result is either *head* (H) or *tail* (T).
At time t_1: stock price will be either $S_1(H)$ or $S_1(T)$.
Denote $u = \frac{S_1(H)}{S_0}$ the up-factor and $d = \frac{S_1(T)}{S_0}$ the down-factor.
Assume $d < u$ (if $d > u$ relabel; if $d = u$ then S_1 not random), it is common to have $d = 1/u$.



Arbitrage

Let r be the interest rate at the money market. Assume $r \geq 0$ and same for

$$\text{investing } 1 \text{ EUR at } t_0 \longrightarrow (1+r) \text{ EUR at } t_1,$$
$$\text{borrowing } 1 \text{ EUR at } t_0 \longrightarrow \text{debt } (1+r) \text{ EUR at } t_1.$$

Arbitrage = a trading strategy that begins with *no money*, has zero probability of losing money, and has a positive probability of making money.

Lemma

No arbitrage if and only if $0 < d < 1 + r < u$.

Derivative securities

European call option $=$ the right (but not the obligation) to buy one share of the stock at time one for the strike price K.
Assume: $S_1(T) < K < S_1(H)$.

$T \Rightarrow$ option expires worthless,

$H \Rightarrow$ option can be exercised, yields profit $S_1(H) - K$.

The option at time one is worth $(S_1 - K)^+ = \max\{0, S_1 - K\}$.
European put option pays off $(K - S_1)^+$.
Both are derivative securities, pay either $V_1(H)$ or $V_1(T)$.

Fundamental question: How much is it worth at time zero?

Assumptions

1. Shares of stock can be subdivided for sale and purchase (exist lots of options).
2. The interest rate is the same for investing and borrowing (close to be true for large institutions).
3. The purchase price = the selling price, i.e. the bid-ask spread is zero (NOT satisfied in practice, not trivial).
4. At any time, the stock can take only two possible values in the next period (binomial model). or the stock price is a geometric Brownian motion (continuous-time model) that leads to Black-Scholes-Merton model (this assumption is empirically NOT true).

Problem: Find V_0

- At t_0: initial wealth X_0, we buy Δ_0 shares of stock, our cash position is $X_0 - \Delta_0 S_0$,
- At t_1:

$$\begin{aligned} X_1 &= \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) \\ &= (1+r)X_0 + \Delta_0[S_1 - (1+r)S_0] \end{aligned}$$

- Choose X_0 and Δ_0 so that $X_1(H) = V_1(H)$ and $X_1(T) = V_1(T)$:

$$\begin{aligned} V_1(H) &= (1+r)X_0 + \Delta_0[S_1(H) - (1+r)S_0] \\ V_1(T) &= (1+r)X_0 + \Delta_0[S_1(T) - (1+r)S_0]. \end{aligned}$$

- Solution:

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$$ - delta hedging formula

$$X_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)] =: V_0$$ - we hedged a short position,

where $\tilde{p} = \frac{1+r-d}{u-d}$ and $\tilde{q} = 1 - \tilde{p} = \frac{u-1-r}{u-d}$ are risk neutral probabilities

Example: $r = 0.25, S_0 = 4, K = 5$

- At t_0: $X_0 = 1.20$, we buy $\Delta_0 = 0.5$ shares of stock for $\Delta_0 S_0 = 2$, i.e. we borrow 0.80 to do so,
our cash position: $X_0 - \Delta_0 S_0 = -0.80$ (i.e. debt),
- At t_1: cash position: $(1+r)(X_0 - \Delta_0 S_0) = -1$ (i.e. grater debt), our portfolio will be

$$\text{either } X_1(H) = \Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0) = 4 - 1 = 3$$
$$\text{or } X_1(T) = \Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0) = 1 - 1 = 0.$$

value of the option is

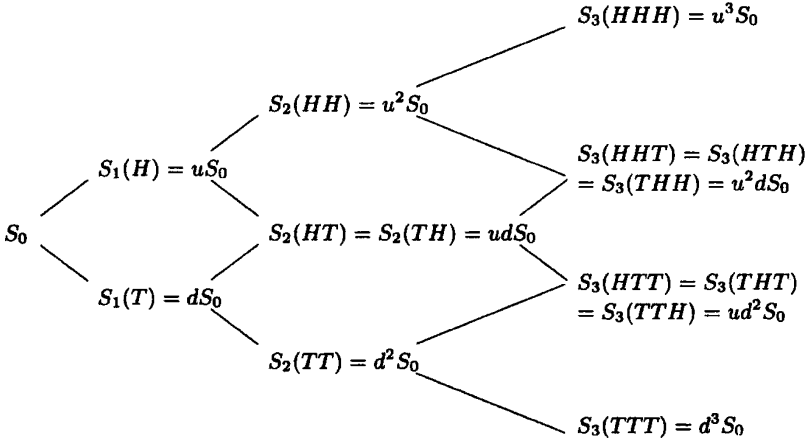
$$\text{either } V_1(H) = (S_1(H) - K)^+ = (8-5)^+ = 3$$
$$\text{or } V_1(T) = (S_1(T) - K)^+ = (2-5)^+ = 0.$$

We have replicated the option by trading in the stock and money market. Here $\tilde{p} = \tilde{q} = 1/2$ and the no-arbitrage price

$$V_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(t)] = \frac{2}{5}[3+0] = \frac{6}{5} = 1.20.$$

Multi-Period Binomial Pricing Model

For example general three period model:



Consider N coin tosses $\omega_1, \omega_2, \dots, \omega_N$.
Now Δ_n can be different in each time t_n.

Replicating in N-Period Binomial Pricing Model

Theorem

Let $0 < d < 1 + r < u, \tilde{p} = \frac{1+r-d}{u-d}, \tilde{q} = \frac{u-1-r}{u-d}$. Let V_N be (a derivative security paying off at time N) a random variable. Define recursively backward in time for $n = N-1, N-2, \dots, 1, 0$ values V_n and Δ_n by

$$V_n = \frac{1}{1+r}[\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)],$$

$$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} = \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n}.$$

Define recursively forward

$$X_0 = V_0,$$

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n).$$

Then $X_N = V_N$ for all possible coin tosses outcomes $\omega_1, \dots, \omega_N$.

Symmetric random walk

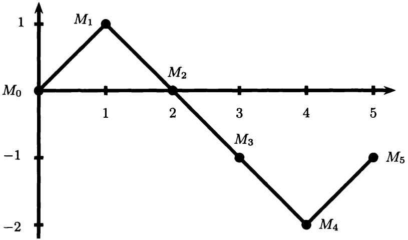
Consider a fair coin ($p = q = 1/2$). For $j = 1, 2, \ldots$ let

$$X_j = \begin{cases} 1 & \text{if } \omega_j = H, \\ -1 & \text{if } \omega_j = T. \end{cases}$$

Define a symmetric random walk $M_n, n = 0, 1, 2, \ldots$ by

$$M_0 = 0 \text{ and } M_n = \sum_{j=1}^{n} X_j, n = 1, 2, \ldots.$$

An example of five steps random walk:



For $u = 2$ and $d = \frac{1}{2}$ and S_0 given, we may write $S_n = S_0 \cdot 2^{M_n}$.

Properties of symmetric random walk:

Properties of X_j: $\mathbb{E}[X_j] = 0, \mathrm{Var}[X_j] = 1$, for all j.
Properties of M_n:

- independent increments: for any $0 = n_0 < n_1 < \cdots < n_m$, the random variables

$$(M_{n_1} - M_{n_0}), (M_{n_2} - M_{n_1}), \ldots, (M_{n_m} - M_{n_{m-1}})$$

  are independent and

$$\mathbb{E}[M_{n_{i+1}} - M_{n_i}] = 0,$$
$$\mathrm{Var}[M_{n_{i+1}} - M_{n_i}] = n_{i+1} - n_i.$$

- M_n is Markov process (memory less) and martingale (no tendency to rise or fall),
- quadratic variation:

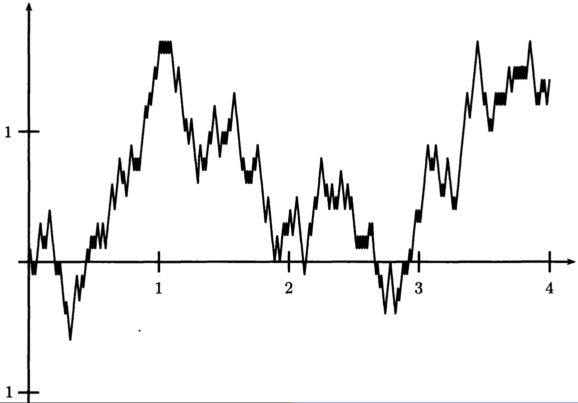
$$\sum_{j=1}^{n} [M_j - M_{j-1}]^2 = n.$$

Scaled symmetric random walk

Fix a positive integer n and define

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt},$$

provided nt is itself an integer (if not then linearly interpolate).
A sample path of $W^{(100)}$:



Properties of scaled symmetric random walk

Properties of $W^{(n)}(t)$:

- independent increments: for any $0 = t_0 < t_1 < \cdots < t_m$ such that $nt_j \in \mathbf{N}$, the random variables

$$(W^{(n)}(t_1) - W^{(n)}(t_0)), \ldots, (W^{(n)}(t_m) - W^{(n)}(t_{m-1}))$$

  are independent and for $0 \leq s \leq t$ and $ns, nt \in \mathbf{N}$

$$\mathbb{E}[W^{(n)}(t) - W^{(n)}(s)] = 0,$$
$$\mathrm{Var}[W^{(n)}(t) - W^{(n)}(s)] = t - s.$$

- $W^{(n)}(t)$ is Markov process and martingale,
- quadratic variation:

$$\sum_{j=1}^{nt} \left[W^{(n)}\left(\frac{j}{n}\right) - W^{(n)}\left(\frac{j-1}{n}\right)\right]^2 = t.$$

Central Limit Theorem

Theorem

Fix $t \geq 0$. As $n \to +\infty$, the distribution of the scaled random walk $W^{(n)}(t)$ evaluated at time t converges to the normal distribution with mean zero and variance t.

Scaled random walk $W^{(n)}(t)$ approximates a Brownian motion. Binomial model is a discrete-time version of the geometric Brownian motion which is the basis for the Black-Scholes-Merton option pricing formula.

Lognormal distribution as the limit of the binomial model

Consider a Binomial model on $[0, t]$, n steps per unit time $(nt \in \mathbf{N})$. Let

- up factor $u_n = 1 + \frac{\sigma}{\sqrt{n}}$, down factor $d_n = 1 - \frac{\sigma}{\sqrt{n}}$, where $\sigma > 0$ is the volatility parameter,
- interest rate be zero: $r = 0$,
- risk neutral probabilities be $\tilde{p} = \frac{1+r-d_n}{u_n-d_n} = \frac{1}{2}$ and $\tilde{q} = \frac{u_n-1-r}{u_n-d_n} = \frac{1}{2}$,
- H_{nt}, T_{nt} be the number of H, T in the first nt coin tosses,

$$H_{nt} + T_{nt} = nt.$$

- random walk $M_{nt} = H_{nt} - T_{nt}$ and hence

$$H_{nt} = \frac{1}{2}(nt + M_{nt}) \quad \text{and} \quad T_{nt} = \frac{1}{2}(nt - M_{nt}).$$

Then

$$\begin{aligned} S_n(t) &= S(0) u_n^{H_{nt}} d_n^{T_{nt}} \\ &= S(0) \left(1 + \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt+M_{nt})} \left(1 - \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt-M_{nt})}. \end{aligned}$$

Lognormal distribution as the limit of the binomial model

Theorem

As $n \to +\infty$, the distribution of $S_n(t)$ converges to the distribution of

$$S(t) = S(0) \exp\left\{\sigma W(t) - \frac{1}{2}\sigma^2 t\right\},$$

where $W(t)$ is a normal random variable with zero mean and variance t.

Note that $X(t) = \sigma W(t) - \frac{1}{2}\sigma^2 t$ is a normal random variable with

$$\mathbb{E}[X(t)] = \sigma\mathbb{E}[W(t)] - \frac{1}{2}\sigma^2 t = -\frac{1}{2}\sigma^2 t,$$

$$\mathrm{Var}[X(t)] = \mathbb{E}[X(t) - \mathbb{E}[X(t)]]^2 = \mathbb{E}[\sigma W(t)]^2 = \sigma^2\mathbb{E}[W(t)]^2 = \sigma^2 t.$$

Brownian motion (Wiener process)

Brownian motion is random drifting of particles suspended in a fluid or gas.
Albert Einstein: Annus Mirabilis (1905) paper about "stochastic model of Brownian motion" (Nobel Prize 1921),

A. Einstein: *On the movement of small particles suspended in a stationary liquid demanded by the molecular-kinetic theory of heat*, Ann. Phys. 17.

Wiener process is mathematical description of Brownian motion:
$W : [0, \infty) \times \Omega \to \mathbf{R}$

- $W(0) = 0$ a.s. (with prob. 1),
- for all $0 = t_0 < t_1 < \dots t_m$ the increments

$$W(t_1) - W(t_0), W(t_2) - W(t_1), \dots, W(t_m) - W(t_{m-1})$$

are independent normally distributed random variables with

$$\mathbb{E}[W(t_{i+1}) - W(t_i)] = 0$$
$$\mathrm{Var}[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i.$$

Properties of Wiener process

- For all $0 \leq s \leq t$: $W(t) - W(s) \sim \mathcal{N}(0, t-s)$, i.e.

$$\mathbb{E}[W(t) - W(s)] = 0,$$
$$\mathbb{E}[W(t) - W(s)]^2 = t - s.$$

- $W(t)$ has continuous paths that are NOWHERE differentiable ("infinitely fast" coin tossing).
- For all $0 \leq s \leq t$: $\mathrm{Cov}(W(s), W(t)) = \mathbb{E}[W(s)W(t)] = s$.
- $W(t)$ is Markov process and martingale.
- Let D_n be a partition of the interval $[0, T]$:
  $0 = t_0 < t_1 < \cdots < t_n = T$, and
  $||D_n|| := \max_{0 \leq k \leq n-1} (t_{k+1} - t_k)$. Then quadratic variation:

$$[W, W](T) = \lim_{||D_n|| \to 0} \sum_{j=0}^{n-1} [W(t_{j+1}) - W(t_j)]^2 = T.$$

Formally we write $dW(t)dW(t) = dt$.

Properties of Wiener process

- Note that $dW(t)\,dt = 0$, i.e.

$$\lim_{||D_n||\to 0} \sum_{j=0}^{n-1} \underbrace{[W(t_{j+1}) - W(t_j)]}_{\substack{\leq \\ \max_{0\leq k\leq n-1} |W(t_{k+1})-W(t_k)|}} [t_{j+1} - t_j] = 0,$$

- $dt\,dt = 0$, i.e.

$$\lim_{||D_n||\to 0} \sum_{j=0}^{n-1} \underbrace{[t_{j+1} - t_j]}_{\substack{\leq \\ \max_{0\leq k\leq n-1} [t_{k+1}-t_k]}} [t_{j+1} - t_j] = \lim_{||D_n||\to 0} ||D_n|| \cdot T = 0.$$

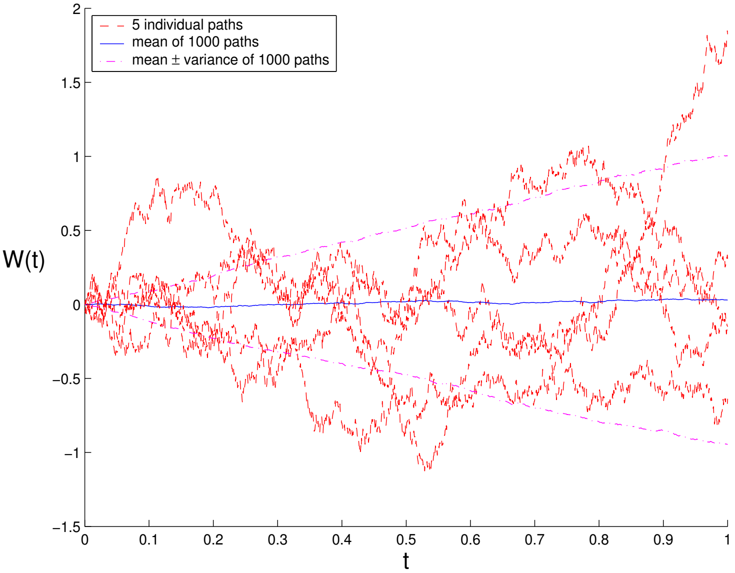
Numerical simulation of Wiener process

- Consider discretized Wiener process, $W(t)$ is specified at discrete values of t.
- For equidistant discretization $\delta t = T/N$, N some positive integer.
- Algorithm:

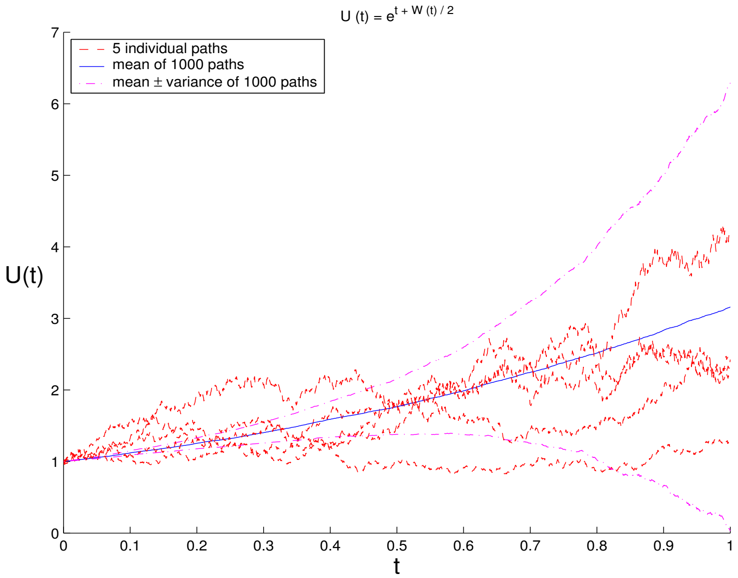
$$
\begin{aligned}
W_0 &= 0, \\
W_j &= W_{j-1} + dW_j, \quad j = 1, 2, \ldots, N,
\end{aligned}
$$

  where $W_j = W(t_j), t_j = j\delta t$ and dW_j is an independent RV of the form $\sqrt{\delta t}\,\mathcal{N}(0,1)$
- It is easy to implement.
- We can simulate a function $u(t) = u(W(t))$ along Wiener paths.

Discretized Paths of Wiener process



Function of Wiener process



Geometric Brownian Motion

Let α and $\sigma > 0$ be constants. Then geometric Brownian motion is process

$$S(t) = S(0) \exp\left\{ \sigma W(t) + \left(\alpha - \frac{1}{2}\sigma^2 \right) t \right\}.$$

This is the asset-price model used in the Black-Scholes-Merton option-pricing formula.

Volatility parameter estimate

- Say we observe $S(t)$ on a time interval $[T_1, T_2]$.
- Choose a partition $T_1 = t_0 < t_1 < \cdots < t_m = T_2$.
- Log returns on $[t_j, t_{j+1}]$:

$$\ln \frac{S(t_{j+1})}{S(t_j)} = \sigma[W(t_{j+1}) - W(t_j)] + \left(\alpha - \frac{1}{2}\sigma^2\right)[t_{j+1} - t_j].$$

- Realized volatility on $[T_1, T_2]$:

$$\begin{aligned}\sum_{j=0}^{m-1}\left[\ln \frac{S(t_{j+1})}{S(t_j)}\right]^2 &= \sigma^2 \sum_{j=0}^{m-1}\left[W(t_{j+1}) - W(t_j)\right]^2 \\ &+ \left(\alpha - \frac{1}{2}\sigma^2\right)\sum_{j=0}^{m-1}\left[t_{j+1} - t_j\right]^2 \\ &+ 2\sigma\left(\alpha - \frac{1}{2}\sigma^2\right)\sum_{j=0}^{m-1}\left[W(t_{j+1}) - W(t_j)\right]\left[t_{j+1} - t_j\right]\end{aligned}$$

$$\sigma^2 \approx \frac{1}{T_2 - T_1}\sum_{j=0}^{m-1}\left[\ln \frac{S(t_{j+1})}{S(t_j)}\right]^2, \quad \text{provided } ||D_n|| \text{ is small.}$$

Probability Theory Preliminaries

- Probability space $(\Omega, \mathcal{F}, \mathbb{P})$
  - Ω any set, *state space*; $\omega \in \Omega$ a sample point
  - $\mathcal{F}$ a σ-algebra of subsets of Ω; $A \in \mathcal{F}$ an event
  - $\mathbb{P}$ a probability measure on $\mathcal{F}$; $\mathbb{P}(A)$ a probability of event A
- Random Variables
  - random variable $X : \Omega \to \mathbf{R}$ on $(\Omega, \mathcal{F}, \mathbb{P})$
  - realization sample $X(\omega)$
  - X measurable if $X^{-1}(a) := \{\omega \in \Omega; X(\omega) \leq a\} \in \mathcal{F}, \forall a \in \mathbf{R}$
  - distribution function $F_x : \mathbf{R} \to [0, 1]$; $F_x(a) := \mathbb{P}(X^{-1}(a))$
  - continuous vs. discrete random variables

Probability Theory Preliminaries

- Moments of Random Variables
  - expectation, expected value $\mathbb{E}(X) := \int_\Omega X \, d\mathbb{P}$
  - p-th moment $\mathbb{E}(X^p) := \int_\Omega |X|^p \, d\mathbb{P}$
  - variance $\mathrm{Var}(X) := \mathbb{E}\big(|X - \mathbb{E}(X)|^2\big) = \mathbb{E}(|X|^2) - |\mathbb{E}(X)|^2$
- Independence of RVs $X_1, X_2, \ldots, X_n, \ldots$
- Convergence of RVs $X_n \to \bar{X}$ as $n \to \infty$
  - with probability 1: $X_n(\omega) \to \bar{X}, \forall \omega \in \Omega$
  - in p-th moment: $\mathbb{E}(|X_n - \bar{X}|^p) \to 0$
  - in probability: $\mathbb{P}(\{\omega \in \Omega; |X_n(\omega) - \bar{X}(\omega)| \geq \varepsilon\}) \to 0, \forall \varepsilon > 0$
  - in distribution: $F_{X_n}(a) \to F_{\bar{X}}(a), \forall a \in \mathbf{R}$

Stochastic processes

Stochastic process is a parametrized collection of random variables:

- $X : \mathbf{T} \times \Omega \to \mathbf{R}$, where $\mathbf{T} \subseteq \mathbf{R}$ is a time set
- X is a stochastic process if $X_t : \Omega \to \mathbf{R}$ is a random variable for each $t \in \mathbf{T}$
- sample path realization $X_{\cdot}(\omega) : \mathbf{T} \to \mathbf{R}$, ω fixed
- many possible types of time dependence
  - independent: X_t, X_s if $t \neq s$
  - identically distributed: $F_{X_t}(x) \equiv F(x), \forall t \in \mathbf{T}$
  - independent increments: $X_{\tau_2} - X_{\tau_1}, X_{\tau_4} - X_{\tau_3}, \ldots$
  - Markovian: future depends only on present (not both present and past)

Diffusion processes

- $X : [0, T] \times \Omega \to \mathbf{R}$, $\forall s \in [0, T], x \in \mathbf{R}, \varepsilon > 0$,
  $\int_B p(s, x; t, y) dy = \mathbb{P}(\{\omega; X_t(\omega) \in B | X_s = x\})$:
  - $\lim\limits_{t \downarrow s} \frac{1}{t-s} \int\limits_{|y-x|>\varepsilon} p(s, x; t, y) dy = 0$ ... no jumps
  - $\lim\limits_{t \downarrow s} \frac{1}{t-s} \int\limits_{|y-x|<\varepsilon} (y - x) p(s, x; t, y) dy = a(s, x)$ ... drift
  - $\lim\limits_{t \downarrow s} \frac{1}{t-s} \int\limits_{|y-x|<\varepsilon} (y - x)^2 p(s, x; t, y) dy = b^2(s, x)$ ... squared diffusion coefficients
- Markovian, sample path continuous, transition densities $p(s, x; t, \cdot)$ satisfy Kolmogorov PDEs

Standard Wiener process (Standard Brownian motion)

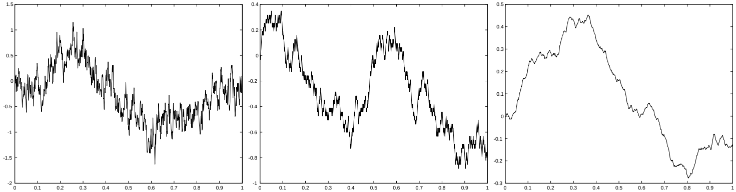
- Simplest, prototype diffusion process describing physically observed phenomenon
- Standard Wiener process: $W : [0, \infty) \times \Omega \to \mathbf{R}$
  - $W_0 = 0$ w.p.1
  - $W_t - W_s \sim \mathcal{N}(0, t - s)$, i.e.

$$\mathbb{E}[W_t - W_s] = 0,$$
$$\mathbb{E}\left[(W_t - W_s)^2\right] = t - s$$

  - independent increments $W_{t_2} - W_{t_1}, W_{t_4} - W_{t_3}, \dots$
  - sample path continuous, but NOT differentiable anywhere

Fractional Brownian motion

- Generalization of Brownian motion (B. Mandelbrott, V. Ness)
- Hurst parameter $H \in (0, 1)$ (for $H = 1/2$: BM)
- $\mathbb{E}\beta_t^H \beta_s^H = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} + |t - s|^{2H}\right)$, $\forall t, s \in \mathbf{R}$
- for $H > 1/2$ positively correlated, for $H < 1/2$ negatively correlated,
- Paths for H equal to 0.25, 0.5 and 0.75:



Deterministic integrals

- Riemann-Stieltjes integral

$$\int_0^T f(t)dR(t) = \lim_{N\to\infty} \sum_{j=0}^{N-1} f(\tau_j)[R(t_{j+1}) - R(t_j)]$$

  exists iff R has bounded variation on $[0, T]$, $\tau_j \in [t_j, t_{j+1}]$.
- A stochastic integral cannot be a Riemann-Stieltjes integral for each ω
- we may consider also Lebesgue integrals

Stochastic integrals

- Itô's stochastic integral

$$\int_0^T f(t,\omega)dW_t(\omega) = \lim_{N\to\infty}\sum_{j=0}^{N-1} f(t_j,\omega)[W_{t_{j+1}}(\omega) - W_{t_j}(\omega)]$$

admissible integrands for

  - $\mathbb{E}(f^2(t,\cdot)) < \infty$
  - $f(t,\cdot)$ non-anticipative (indep. of $W_\tau - W_t, \forall \tau > t$)

$f(t_j,\omega)$ evaluated at the beginning of each interval $[t_j, t_{j+1}]$

- Stratonovich stochastic integral $\int\limits_0^T f(s,\omega)\circ dW(t,\omega)$ uses mid-points: $f(\frac{t_j+t_{j+1}}{2},\omega)$ - not used in finance

Example

Approximation of Itô's stochastic integral $\int_0^T W(t)dW(t)$

- Exact solution:

$$\int_0^T W(t)dW(t) = \frac{1}{2}W(T)^2 - \frac{1}{2}T$$

- Approximation:

$$\sum_{j=0}^{N-1} W(t_j)(W(t_{j+1}) - W(t_j))$$
$$= \frac{1}{2}\sum_{j=0}^{N-1}[W(t_j)^2 - W(t_j)^2 - (W(t_{j+1}) - W(t_j))^2]$$
$$= \frac{1}{2}\left(W(T)^2 - W(0)^2 - \underbrace{\sum_{j=0}^{N-1}(W(t_{j+1}) - W(t_j))^2}_{\text{expected value } T\text{,variance of } O(\delta t)} \right)$$

hence for small δt it converges to exact value.

Itô's isometry

Let $f(t,\omega)$ be bounded and let

$$I(t) = \int_0^t f(s,\omega)\, dW_s(\omega)$$

be the Itô's integral. Then

- Itô's isometry:

$$\mathbb{E}[I^2(t)] = \mathbb{E}\left[\int_0^T f^2(t,\omega)\, dt\right],$$

- quadratic variation:

$$[I,I](t) = \left[\int_0^T f^2(t,\omega)\, dt\right].$$

Itô's process

- Let W_t be a one-dimensional Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$. A one-dimensional Itô's process (or stochastic integral) is a stochastic process X_t on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$X_t = X_0 + \int_0^t u(s,\omega)\, ds + \int_0^t v(s,\omega)\, dW_s,$$

where u and v are "nice". In shorter differential form

$$dX_t = u\, dt + v\, dW_t.$$

Itô's formula

Theorem (One dimensional Itô's formula)

Let X_t be an Itô's process and let $g(t,x) \in \mathcal{C}^2([0,\infty) \times \mathbf{R})$. Then $Y_t = g(t, X_t)$ is also an Itô's process and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)\, dt + \frac{\partial g}{\partial x}(t, X_t)\, dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2,$$

where $(dX_t)^2 = (dX_t)(dX_t)$ is computed according to the rules

$$dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0, \quad dW_t \cdot dW_t = dt.$$

Example: Itô's formula

What is $\int_0^T W(t)dW(t)$?

Choose $X_t = W_t$ and $g(t,x) = \frac{1}{2}x^2$. Then $Y_t = g(t, W_t) = \frac{1}{2}W_t^2$ and by Itô's formula

$$\begin{aligned} dY_t &= \frac{\partial g}{\partial t}\,dt + \frac{\partial g}{\partial x}\,dW_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dW_t)^2 \\ &= W_t\,dW_t + \frac{1}{2}(dW_t)^2 \\ &= W_t\,dW_t + \frac{1}{2}\,dt \\ d\left(\frac{1}{2}W_t^2\right) &= W_t\,dW_t + \frac{1}{2}\,dt. \end{aligned}$$

In other words:

$$\int_0^T W(t)dW(t) = \frac{1}{2}W(T)^2 - \frac{1}{2}T.$$

Integration by parts

What is $\int_0^T t\,dW_t$?
Choose $X_t = W_t$ and $g(t,x) = t \cdot x$. Then $Y_t = g(t, W_t) = tW_t$ and by Itô's formula

$$\begin{aligned} dY_t &= W_t\,dt + t\,dW_t + 0 \\ d(tW_t) &= W_t\,dt + t\,dW_t \\ TW_T &= \int_0^T W_t\,dt + \int_0^T t\,dW_t \\ \int_0^T t\,dW_t &= TW_T - \int_0^T W_t\,dt. \end{aligned}$$

Theorem

Suppose $f(t,\omega)$ is continuous and of bounded variation w.r.t. $t \in [0,T]$ for a.a. ω. Then

$$\int_0^T f(t)\,dW_t = f(T)W_T - \int_0^T W_t\,df_t.$$

Stochastic differential equations

Let $0 \leq t \leq T$; $a, b : [0, T] \times \mathbf{R} \to \mathbf{R}$.

- stochastic differential equation (linear)

$$\begin{aligned} dX_t(\omega) &= a(t, X_t(\omega))dt + b(t, X_t(\omega))dW_t(\omega), \\ X_0(\omega) &= X_0 \end{aligned} \tag{SDE}$$

- or in integral form

$$X_t(\omega) = X_0(\omega) + \\ + \underbrace{\int_0^T a(s, X_s(\omega))ds}_{\substack{\text{deterministic} \\ \text{integral for} \\ \text{each } \omega \in \Omega}} + \underbrace{\int_0^T b(s, X_s(\omega))dW_s(\omega)}_{\substack{\text{stochastic} \\ \text{integral}}}$$

Example: population growth

Simple population growth (or also asset pricing) model

$$\frac{dN}{dt} = a(t)N(t), \quad N(0) = N_0,$$

where $a(t) = r + \alpha\eta_t$, η_t is a white noise, α and r are constant. This equation is equivalent to

$$\begin{aligned} dN_t &= rN_t\,dt + \alpha N_t\,dW_t \\ \int_0^t \frac{dNs}{N_s} &= rt + \alpha W_t. \end{aligned}$$

By Itô's formula $d(\ln N_t) = \frac{1}{N_t}\,dt + \frac{1}{2}\left(-\frac{1}{N_t^2}\right)(dN_t)^2 = \frac{dNt}{N_t} - \frac{1}{2}\alpha^2\,dt$

$$\begin{aligned} \ln\frac{N_t}{N_0} &= (r - \frac{1}{2}\alpha^2)t + \alpha W_t \\ N_t &= N_0 \exp\left((r - \frac{1}{2}\alpha^2)t + \alpha W_t\right). \end{aligned}$$

Existence and uniqueness

Theorem

Let $T > 0$. Suppose that

- *coefficient functions* $a, b : [0, T] \times \mathbf{R} \to \mathbf{R}$, *are continuous and* $\forall t, s \in [0, T]; x, y \in \mathbf{R}$:
  - *lipschitz:* $|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq K_1 |x - y|$
  - *of max. lin. growth:* $|a(t,x)| + |b(t,x)| \leq K_2(1 + |x|)$
  - $|a(s,x) - a(t,x)| + |b(s,x) - b(t,x)| \leq K_3(1 + |x|)|s - t|^{1/2}$
- *initial value* X_0 *is non-anticipative:* $\mathbb{E}\left(|X_0|^2\right) < \infty$.

Then there exists a unique pathwise continuous solution to (SDE) such that

$$\mathbb{E}\left[\int_0^T |X_t|^2 \, dt\right] < \infty.$$

Example: explosion

Equation (deterministic case: $b = 0$)

$$\frac{dX_t}{dt} = X_t^2, \quad X_0 = 1,$$

corresponding to $a(x) = x^2$ (and NOT satisfying the max. lin. growth cond.) has the (unique) solution

$$X_t = \frac{1}{1-t}, \quad 0 \leq t < 1.$$

Thus it is imposible to find a global solution (defined for all t) in this case. We say that in $t = 1$ the solution explodes ($|X_t(\omega)|$ tends to infinity).

Example: uniqueness

Equation (deterministic case: $b = 0$)

$$\frac{dX_t}{dt} = 3X_t^{2/3}, \quad X_0 = 0,$$

has more than one solution. In fact, for any $a > 0$, the function

$$X_t = \begin{cases} 0 & t \le a \\ (t-a)^3 & t > a \end{cases}$$

solves the equation. In this case $a(x) = 3x^{2/3}$ does NOT satisfy the Lipschitz condition at $x = 0$.
Uniqueness means that if $X_1(t,\omega)$ and $X_2(t,\omega)$ are two continuous processes satisfying (SDE), then

$$X_1(t,\omega) = X_2(t,\omega) \quad \text{for all } t \le T, \text{a.s.-}\mathbb{P}.$$

Weak and strong solutions

- Strong solution:

$$X_t(\omega) = X_0(\omega) + \int_0^t a(s, X_s(\omega))\, ds + \int_0^t b(s, X_s(\omega))\, dW_s(\omega)$$

  The version of Wiener process W_t is given in advance.

- If we are only given functions $a(t,x)$ and $b(t,x)$ and ask for a pair of processes $(\tilde{X}_t, \tilde{W}_t)$ on a probability space $(\Omega, \tilde{\mathcal{F}}, \mathbb{P})$ such that

$$d\tilde{X}_t(\omega) = a(t, \tilde{X}_t(\omega))dt + b(t, \tilde{X}_t(\omega))d\tilde{W}_t(\omega),$$

  then the solution $\tilde{X}_t$ (more precisely $(\tilde{X}_t, \tilde{W}_t)$) is called a weak solution - natural concept, it does not specify beforehand the explicit representation of the white noise.

- Strong uniqueness (pathwise) vs. weak uniqueness (identity in law)

Example of weak solution

The Tanaka equation

$$dX_t = \text{sgn}(X_t)\, dW_t, X_0 = 0,$$

does NOT have a strong solution,
but it DOES have a weak solution:
We simply choose X_t to be *any* Wiener process W_t. We define $\tilde{W}_t$ by

$$\tilde{W}_t = \int_0^t \text{sgn}\, W_s\, dW_s = \int_0^t \text{sgn}(X_s)\, dX_s$$

i.e.

$$d\tilde{W}_t = \text{sgn}(X_t)\, dX_t.$$

Then

$$dX_t = \text{sgn}(X_t)\, d\tilde{W}_t,$$

so X_t is a weak solution.

Black-Scholes-Merton Equation

Consider an agent who at time t has a portfolio $X(t)$, holds $\Delta(t)$ shares of stock modelled by geometric Brownian motion:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

and the remainder $X(t) - \Delta(t)S(t)$ invests in the money market with interest rate r (const.). Then

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + r[X(t) - \Delta(t)S(t)]dt \\ &= \Delta(t)[\alpha S(t)dt + \sigma S(t)dW(t)] + r[X(t) - \Delta(t)S(t)]dt \\ &= rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t). \end{aligned}$$

Compare with the discrete model:

$$\begin{aligned} X_{n+1} &= \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n) \\ X_{n+1} - X_n &= \Delta_n(S_{n+1} - S_n) + r(X_n - \Delta_n S_n). \end{aligned}$$

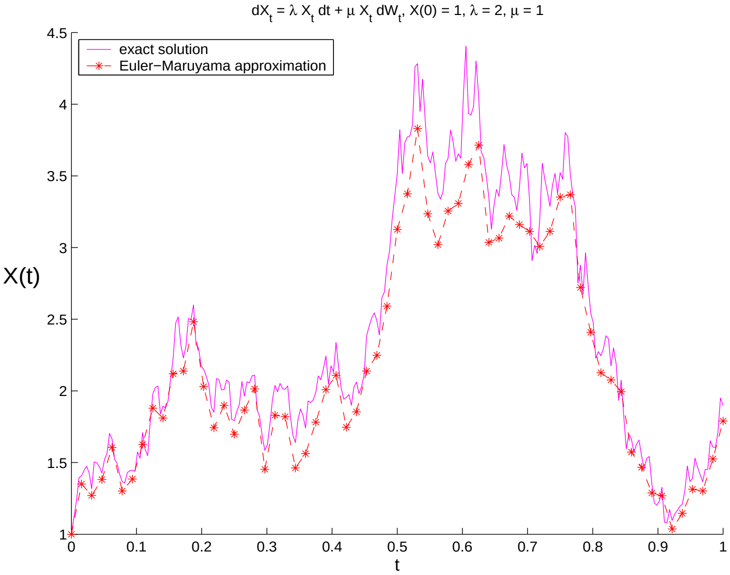
Discounted stock price $e^{-rt}S(t)$ and portfolio $e^{-rt}X(t)$

Differentials of the discounted stock price and portfolio are

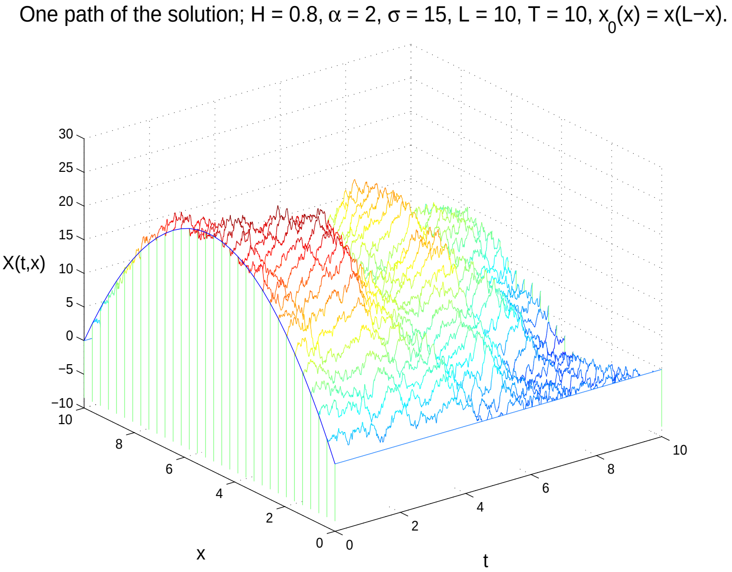
$$
\begin{aligned}
&d(e^{-rt}S(t))\\
&= dg(t,S(t)), \quad \text{where } g(t,x) = e^{-rt}x \text{ and by Itô's formula,}\\
&= g_t(t,S(t))dt + g_x(t,S(t))dS(t) + \frac{1}{2}g_{xx}(t,S(t))dS(t)dS(t),\\
&= -re^{-rt}S(t)dt + e^{-rt}dS(t),\\
&= (\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dW(t),
\end{aligned}
$$

$$
\begin{aligned}
&d(e^{-rt}X(t))\\
&= dg(t,X(t))\\
&= g_t(t,X(t))dt + g_x(t,X(t))dX(t) + \frac{1}{2}g_{xx}(t,X(t))dX(t)dX(t)\\
&= -re^{-rt}X(t)dt + e^{-rt}dX(t)\\
&= \Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t)\\
&= \Delta(t)d(e^{-rt}S(t)).
\end{aligned}
$$

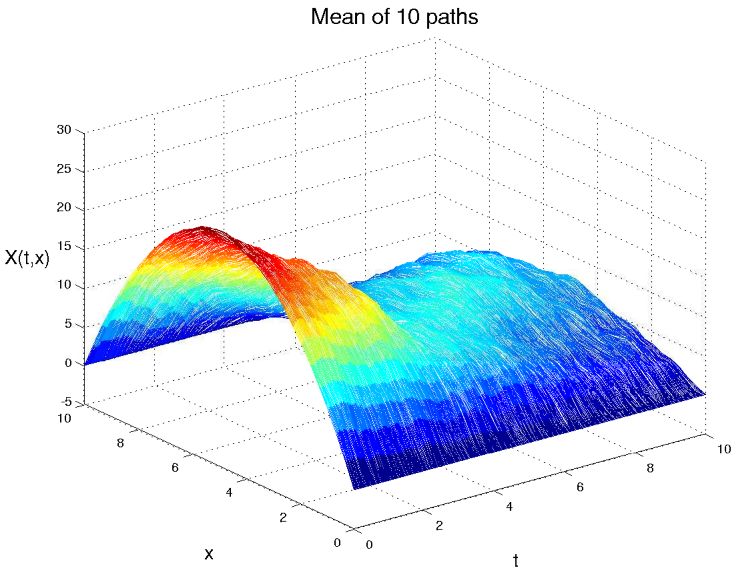
Numerical solution of an SDE



Numerical solution of an SPDE



One path solution to a parabolic equation.



Mean of 10 paths of the solution.

Stochastic differential equations

KMA/USA-A: Introduction to Stochastic Analysis

(4th year, winter term, 2+2, 6 ECTS credits)

- stochastic integal,
- stochastic differential eqations (linear, bilinear),
  - their solution (strong, weak, „mild“),
  - qualitative properties of the solution (limiting behaviour, stability)

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- Markov proceses,
- diffusion and jump processes,
- stochastic differential equations driven by these processes.

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