VIII weekend on Variational Methods and Differential Equations
Università di Catania, Catania (Italy) –
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Two positive solutions to a semilinear spectral problem with a convex/concave nonlinearity of variable $q(x)$-power-type ($q(x)(\geq/\leq)1$)

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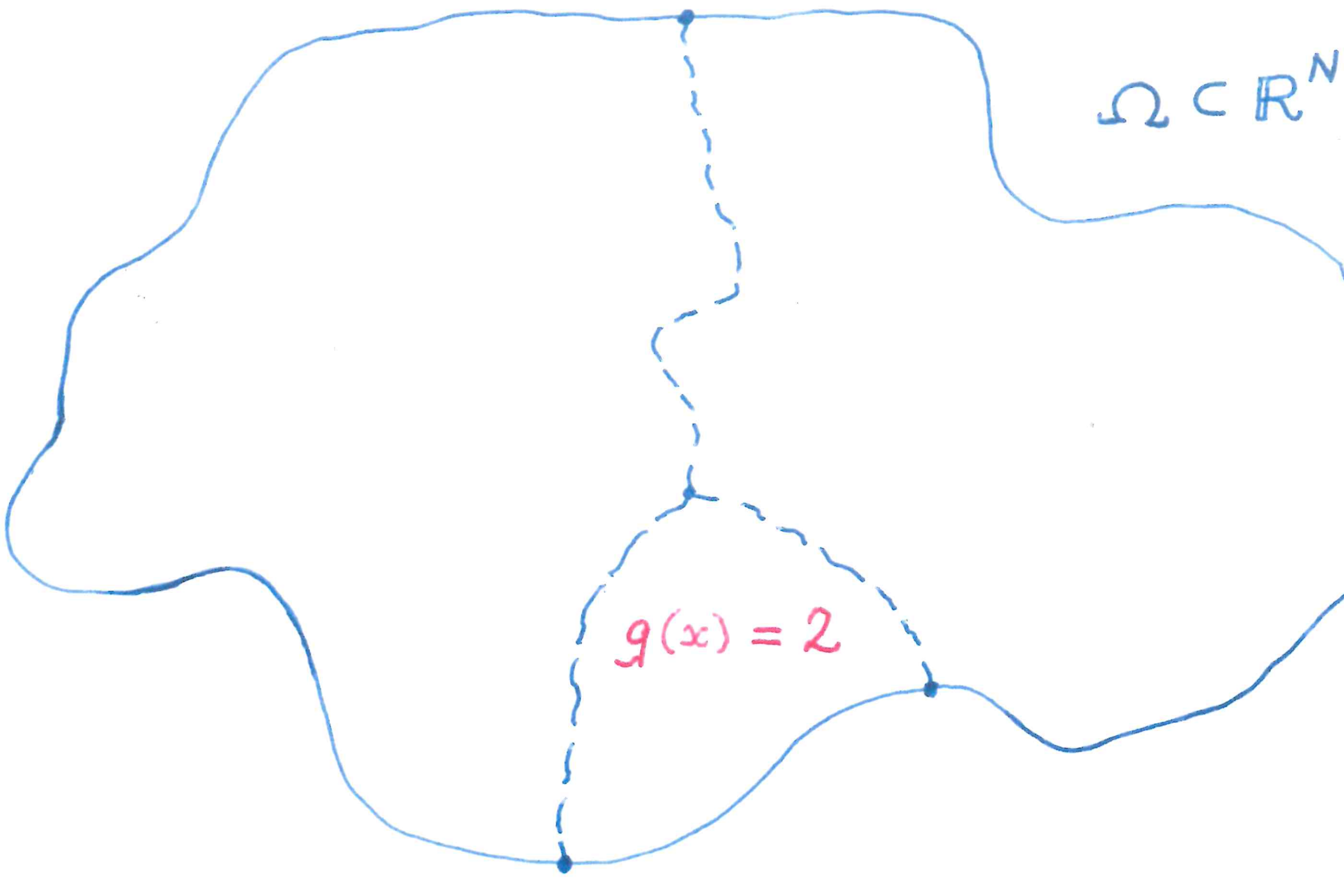
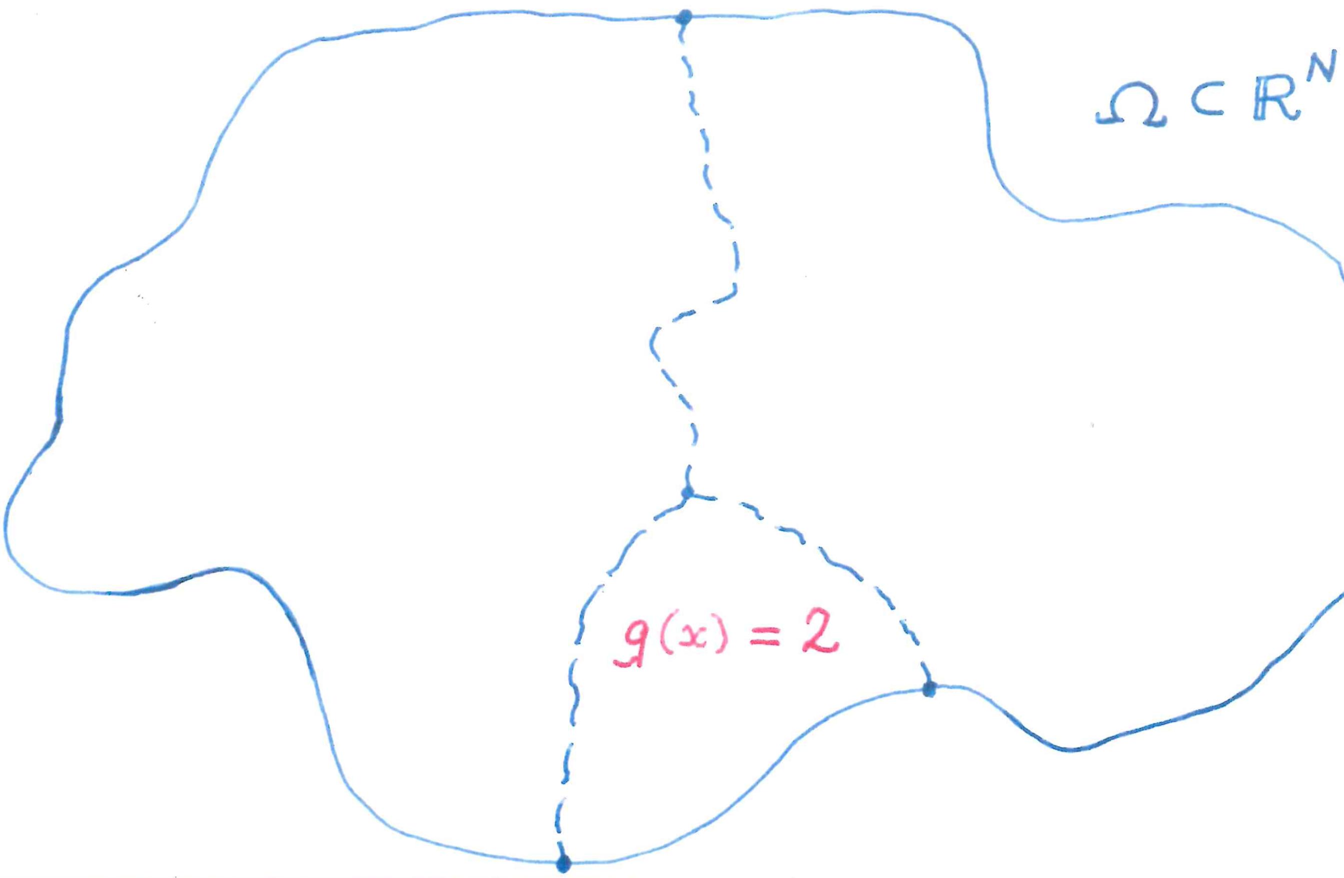


Laplace operator and a nonlinearity of variable $(q(x)-1)$-power-type

$$(1) \qquad \begin{aligned} -\Delta u &= \lambda\,|u(x)|^{q(x)-2}\,u(x) + f(x) \qquad \text{for } x \in \Omega\,; \\ u &= 0 \ \text{ on } \ \partial\Omega\,, \end{aligned}$$

with the **variable exponent** $q : \overline{\Omega} \to \mathbb{R}$ which is assumed to be continuous and ... .
The nonlinearity $s \mapsto s^{q(x)-1} : \mathbb{R}_+ \to \mathbb{R}_+ = [0, \infty)$ is for

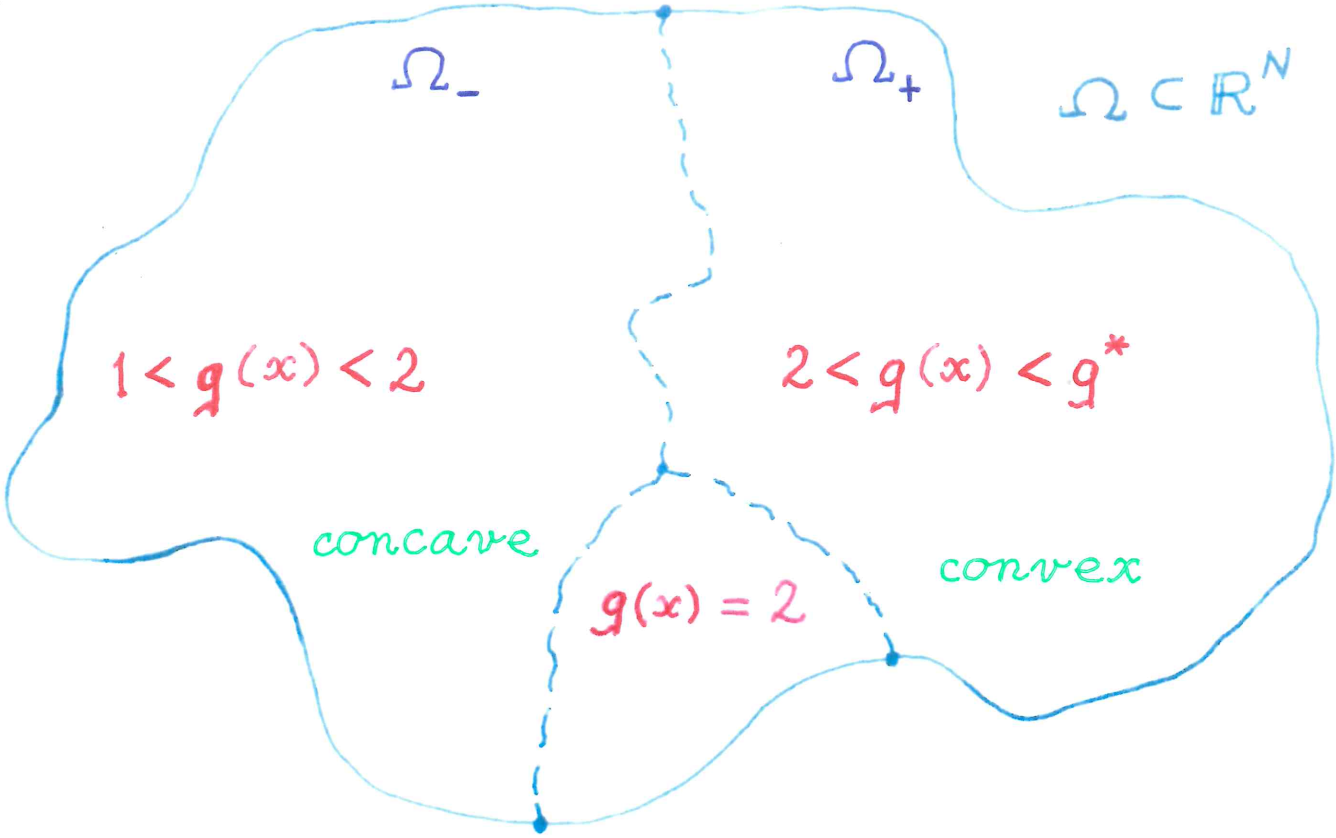
$1 < q(x) \le 2$ **concave**,
$2 \le q(x) < q^* = \frac{N+2}{N-2}$ $(N \ge 3)$ **convex**,
$q^* = +\infty$ if $N = 1, 2$.



Main Hypothesis: $q : \overline{\Omega} \to \mathbb{R}$ is continuous and such that $1 < q(x) < q^*$ and the (open) sets

$$\Omega_- \overset{\text{def}}{=} \{x \in \Omega : q(x) < 2\} \text{ and}$$
$$\Omega_+ \overset{\text{def}}{=} \{x \in \Omega : q(x) > 2\}$$

are **nonempty**.



Laplace operator and a nonlinearity of variable $(q(x)-1)$-power-type

$$(1) \qquad \begin{aligned} -\Delta u &= \lambda\,|u(x)|^{q(x)-2}\,u(x) + h(x) \quad \text{for } x \in \Omega\,; \\ u &= 0 \ \text{ on } \ \partial\Omega\,, \end{aligned}$$

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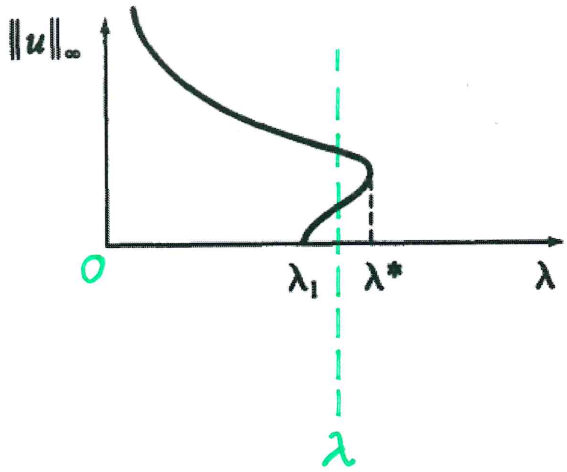
The nonlinearity $s \mapsto s^{q(x)-1} : \mathbb{R}_+ \to \mathbb{R}_+ = [0,\infty)$ is for

$1 < q(x) \leq 2$ **concave**,
$2 \leq q(x) < q^* = \frac{2N}{N-2}$ $(N \geq 3)$ **convex**,
$q^* = +\infty$ if $N = 1,\ 2$.

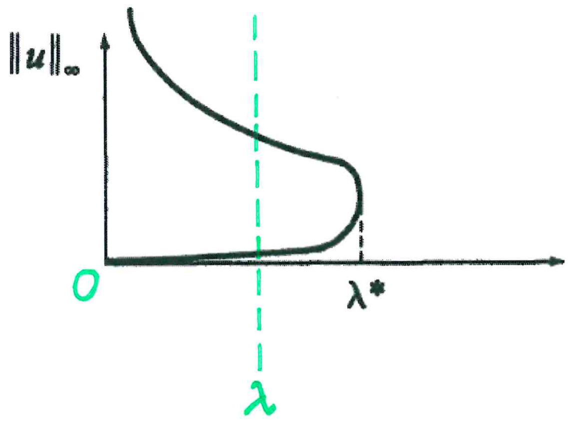
Previous works (e.g. by P.-L. Lions) consider a sum $g(s) = g_1(s) + g_2(s)$ indepedent of $x \in \Omega$ with $g_1 : \mathbb{R}_+ \to \mathbb{R}_+$ **concave** and $g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ **convex**. Under some very natural conditions on the behavior of the functions $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ of $s \in (0, \infty)$ near zero $(s \to 0+)$ and at infinity $(s \to +\infty)$, the following **diagram** is obtained:

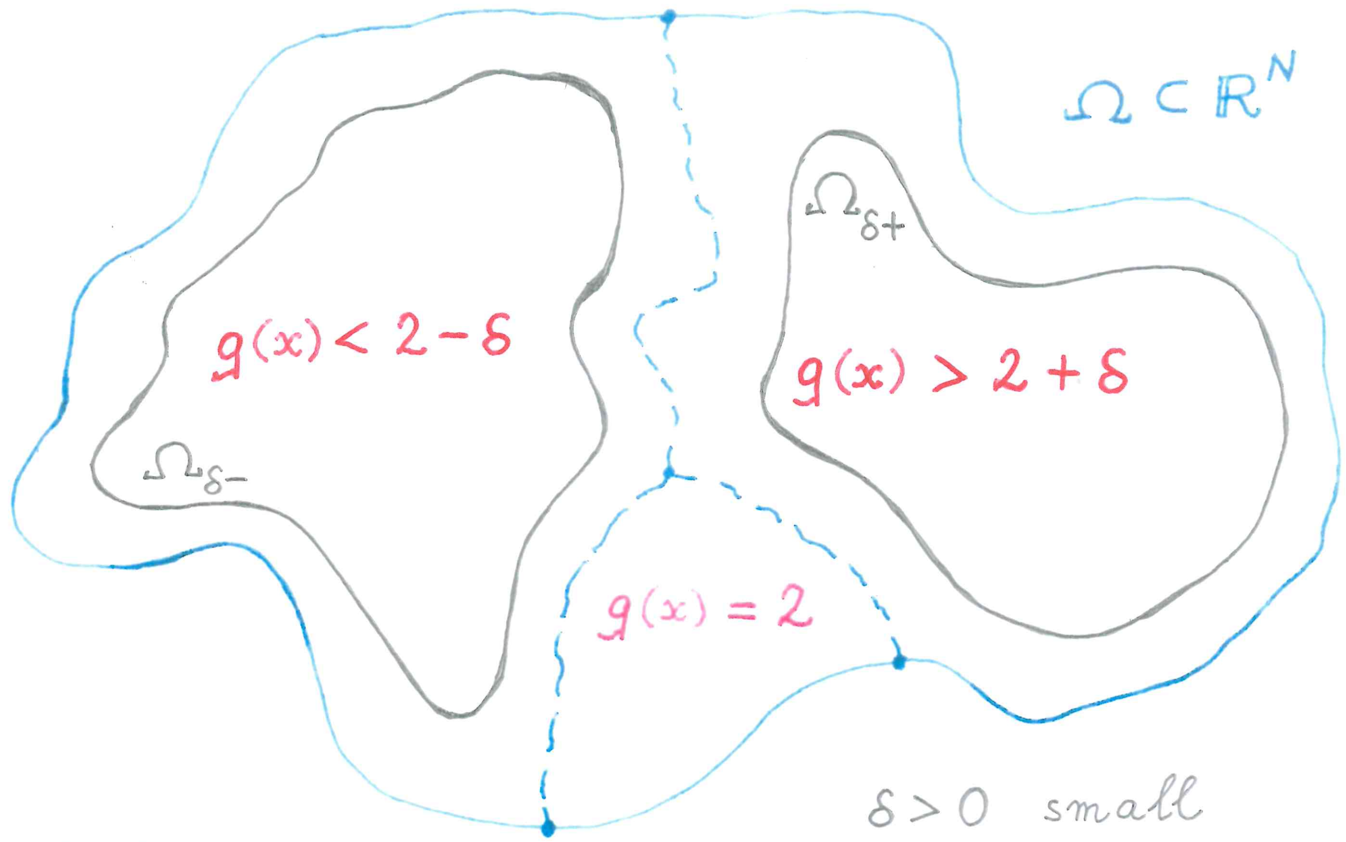
Case 3. $f'(0) = 1, f(t) < t$ for $t > 0$, t small.
Example. $f(t) = t(1 - \sin t) + t^p$ $(1 < p < (N+2)/(N-2))$.



POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

Case 4. $\lim_{t \to 0_+} f(t) t^{-1} = +\infty$.
Example. $f(t) = \sqrt{t} + t^p$ $(1 < p < (N+2)/(N-2))$.





Main Hypothesis: $q : \overline{\Omega} \to \mathbb{R}$ is continuous and such that $1 < q(x) < q^*$ and the (open) sets

$$\Omega_- \stackrel{\text{def}}{=} \{x \in \Omega : q(x) < 2\} \text{ and}$$
$$\Omega_+ \stackrel{\text{def}}{=} \{x \in \Omega : q(x) > 2\}$$

are **nonempty**.

If $\delta > 0$ is small enough, then both sets

$$\Omega_{\delta-} \stackrel{\text{def}}{=} \{x \in \Omega : q(x) < 2 - \delta\} \neq \emptyset \quad \text{and}$$
$$\Omega_{\delta+} \stackrel{\text{def}}{=} \{x \in \Omega : q(x) > 2 + \delta\} \neq \emptyset$$

are **nonempty**.

$$0 < \delta < \min\{2 - q^{(-)}, q^{(+)} - 2\}.$$

(H$_q$)

$$1 < q^{(-)} \stackrel{\text{def}}{=} \min_{\overline{\Omega}} q(x) < p(x) \equiv p = 2$$
$$< q^{(+)} \stackrel{\text{def}}{=} \max_{\overline{\Omega}} q(x) < \infty .$$

We fix $\delta > 0$ small enough, such that both (open) sets

$$\Omega_{\delta-} \stackrel{\text{def}}{=} \{x \in \Omega : \, q(x) < 2 - \delta\} \neq \emptyset \quad \text{and}$$
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are **nonempty**.

Next, we estimate the full nonlinearity

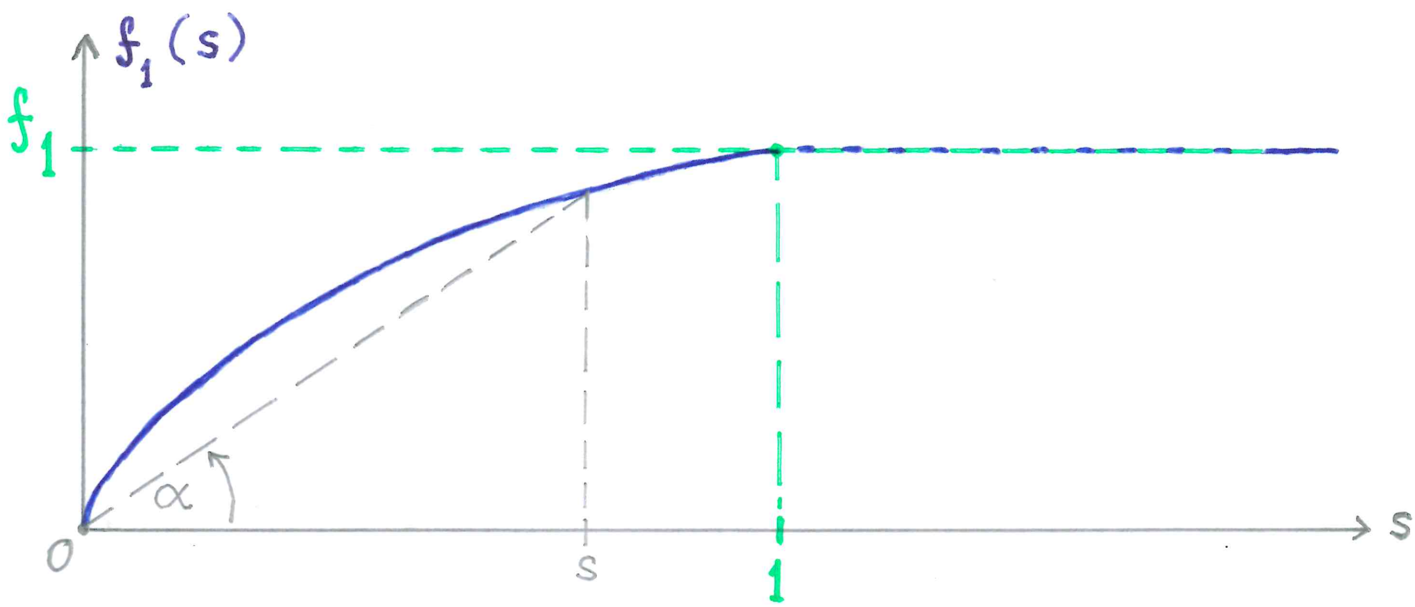
(2) $$f(x,s) \equiv f_\lambda(x,s) = \lambda \, |s|^{q(x)-2} \, s + h(x) \, .$$

from below by a **concave** function, for $(x,s) \in \Omega \times \mathbb{R}_+$:

$$f_1(x,s) \equiv f_{1,\lambda}(x,s) =$$

(3) $$\begin{cases} \lambda \cdot (\min\{s, \, 1\})^{q(x)-1} & \text{if } \; q(x) < 2 - \delta \, ; \\ 0 & \text{if } \; q(x) \geq 2 - \delta \, . \end{cases}$$

Hence, $f_1(x,s) = \lambda \cdot (\min\{s, \, 1\})^{q(x)-1} \cdot \chi_{\Omega_{\delta-}}(x) \, , \; s \geq 0$.

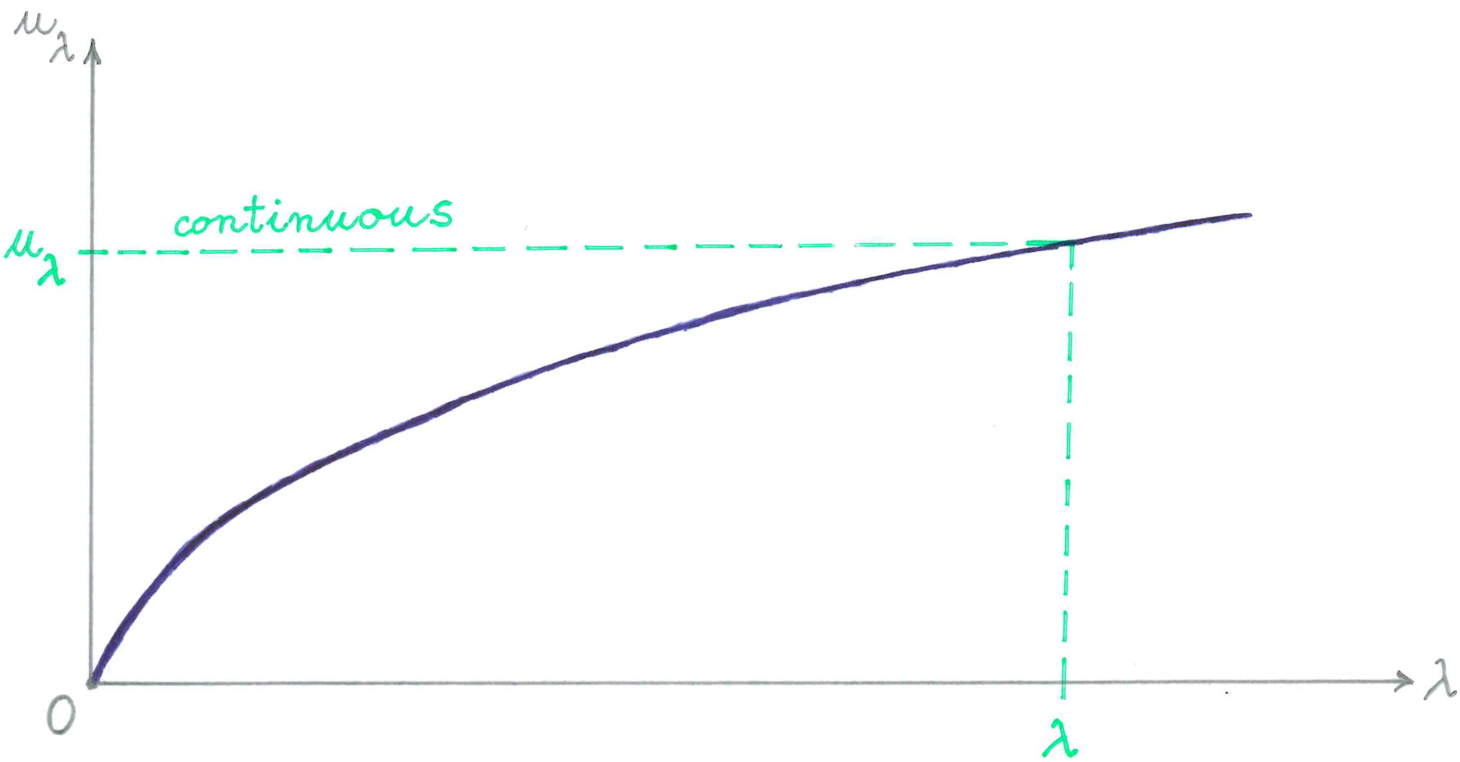


Proposition 1. Given any number $\lambda > 0$ and any nonnegative function $h \in L^\infty(\Omega)$, i.e., $h \in [L^\infty(\Omega)]_+$, the Dirichlet problem

$$(4) \qquad \begin{aligned} -\Delta u &= \lambda f_1(x,s) + h(x) \quad \text{for } x \in \Omega\,; \\ u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned}$$

possesses a unique positive solution $u \equiv u_\lambda \in W_0^{1,2}(\Omega)$.

This result is proved in stability
P. Takáč (NA 1990), P. Hess (Theorem 5.1, 1991 Pitman 1991), motivated by M.A. Krasnosel'skij (1964).



$$\|u_\lambda\|_{C^1(\bar{\Omega})} \le C \equiv \text{const} \equiv C(\alpha, \beta) < \infty\,.$$

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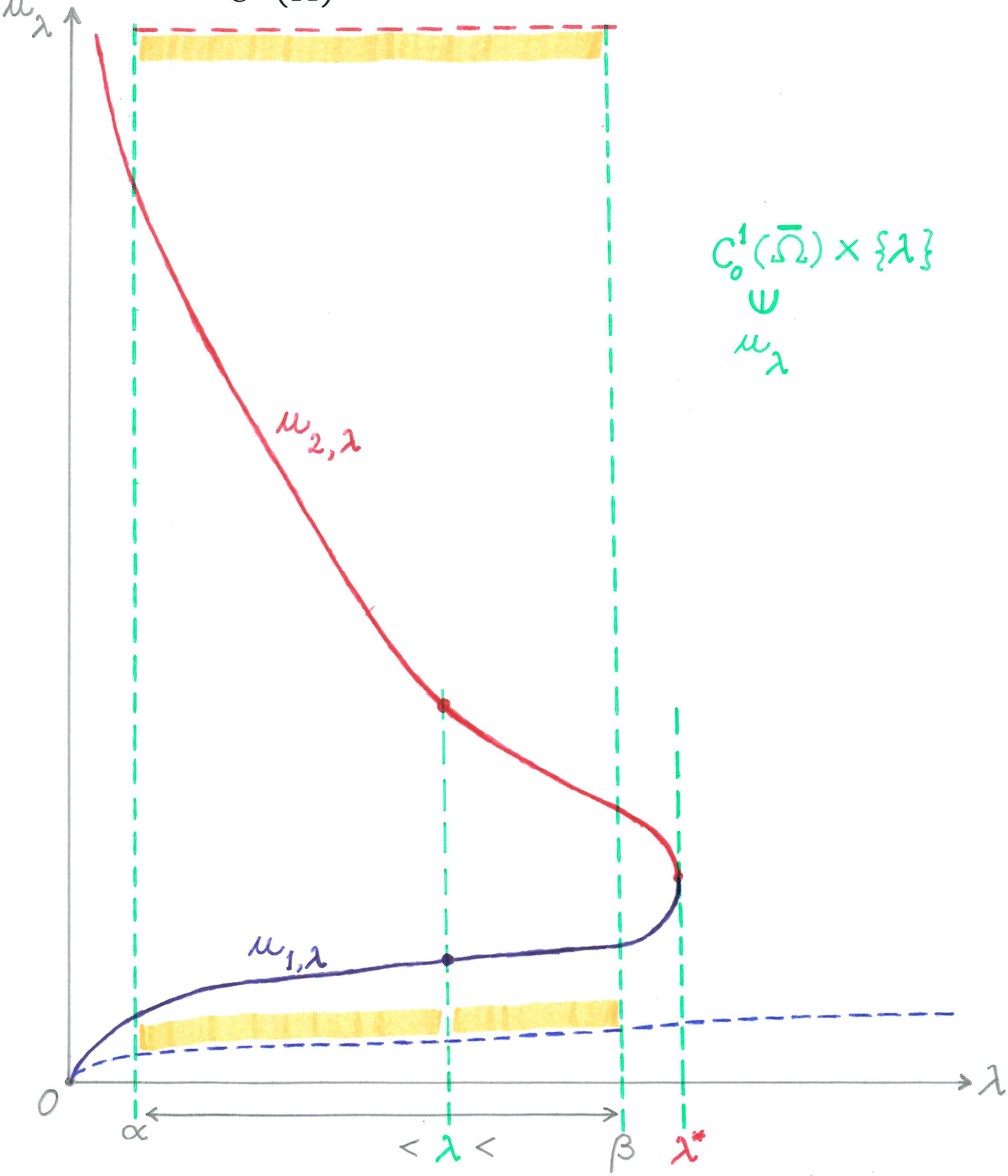


$$\|u_\lambda\|_{C^1(\bar{\Omega})} \le C \equiv \text{const}$$

We begin with the following two a priori results (bounds) on the set $\mathcal{S} \subset \mathbb{R}_+ \times \left[W_0^{1,2}(\Omega)\right]_+$ of all pairs $(\lambda, u) \in \mathbb{R} \times W_0^{1,2}(\Omega)$ such that $\lambda \in \mathbb{R}_+$, $u \in W_0^{1,2}(\Omega)$ is nonnegative in Ω, i.e., $u \in \left[W_0^{1,2}(\Omega)\right]_+$, and

$$(5) \qquad \begin{aligned} -\Delta u &= \lambda\, u(x)^{q(x)-1} + h(x) \quad \text{for } x \in \Omega\,; \\ u &= 0 \ \text{ on } \ \partial\Omega\,. \end{aligned}$$

We assume that $h \in L^\infty(\Omega)$ is a nonnegative function, i.e., $h \in [L^\infty(\Omega)]_+$. We set, for $(x, s) \in \Omega \times \mathbb{R}$.

$$f(x,s) \equiv f_\lambda(x,s) = \lambda\, |s|^{q(x)-2}\, s + h(x)\,.$$

Proposition 2. There exists a number $\lambda^* \geq 0$ such that $\qquad (\lambda, u) \in \mathcal{S} \implies \lambda \leq \lambda^*\,.$
Equivalently, we have

$$\lambda^* \stackrel{\text{def}}{=} \sup_{(\lambda,u)\in\mathcal{S}} \lambda < \infty\,.$$

Proof: Step 1. Consider eq. (5) with $\lambda = \lambda_0 = 0$ and $h = f_0 \in [L^\infty(\Omega)]_+$ supported in

$\Omega_+ \overset{\text{def}}{=} \{x \in \Omega : \, q(x) > 2\}$, $f_0 \not\equiv 0$ in Ω_+, say, $f_0(x) > 0$ for

$$x \in K_\delta \subset \Omega_{\delta +} \overset{\text{def}}{=} \{x \in \Omega : \, q(x) > 2 + \delta\} \,,$$

where K_δ is a compact set with nonempty interior.

Denote the weak solution by $u_0 \in W_0^{1,2}(\Omega)$,

$$u_0(x) \overset{\text{def}}{=} [(-\Delta)^{-1} f_0](x) = \int_\Omega G(x,y)\, f_0(y)\, \mathrm{d}y$$
$$= \int_{\Omega_+} G(x,y)\, f_0(y)\, \mathrm{d}y \quad \text{ for } \; x \in \Omega \,.$$

Hence, by regularity, $u_0 \in C^{1,\theta}(\overline{\Omega})$, $0 < \theta < 1$.
It satisfies the **Hopf maximum principle**:

(HMP) $u_0 > 0$ in Ω and $\frac{\partial u_0}{\partial \boldsymbol{\nu}}(x) < 0$ on $\partial\Omega$.

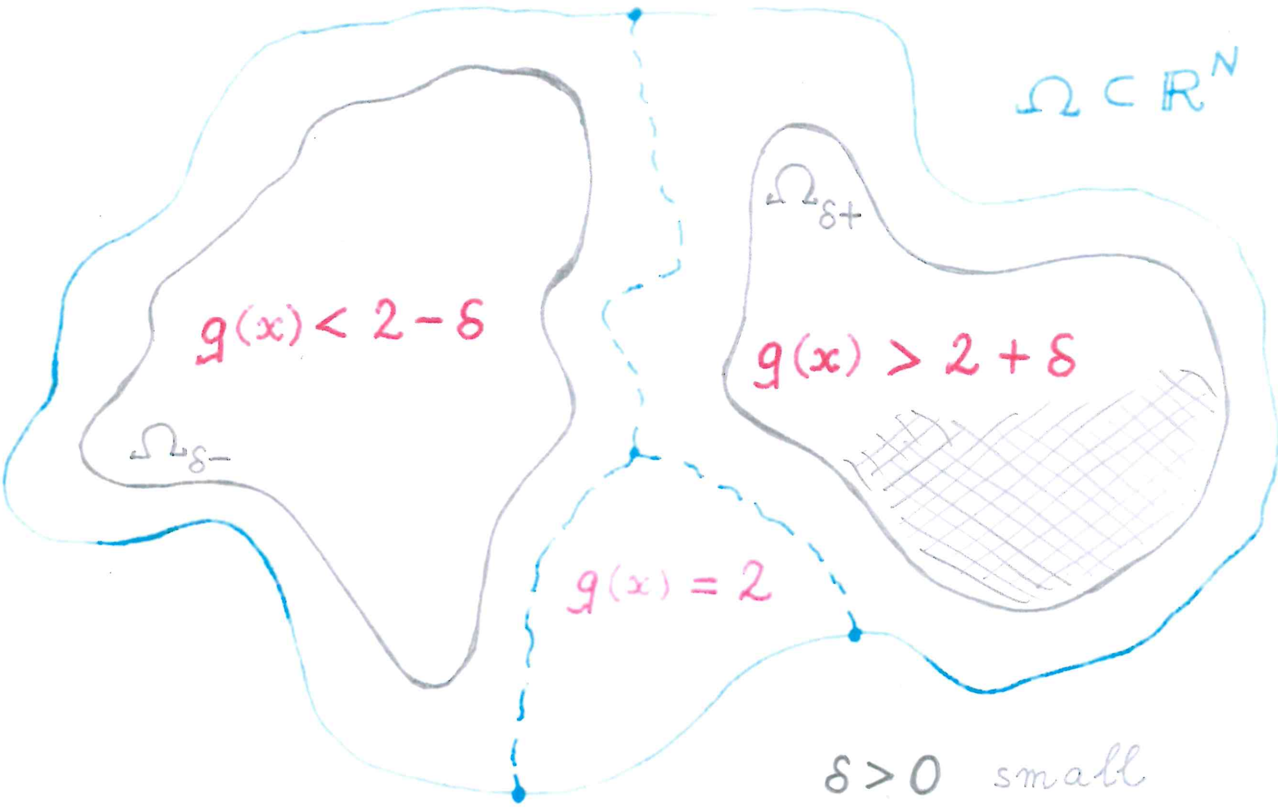
The pair $(\phi, g) = (u_0, f_0)$ verifies
the basic **Poisson equation**

(6) $\quad -\Delta\phi = g(x)$ for $x \in \Omega$; $\quad \phi = 0$ on $\partial\Omega$.

The solution $\phi \in C^{1,\theta}(\overline{\Omega})$, $0 < \theta < 1$, satisfies the **Hopf maximum principle**:

(HMP) $\phi > 0$ in Ω and $\frac{\partial \phi}{\partial \nu}(x) < 0$ on $\partial\Omega$.

Furthermore, $g \in [L^\infty(\Omega)]_+$ is supported in $M_\delta \overset{\text{def}}{=} \{x \in \Omega : q(x) \geq 2 + \delta\} \supset \Omega_{\delta+}$, hence, $g(x) > 0 \implies q(x) \geq 2 + \delta$.



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are **nonempty**.

(H$_q$) $0 < \delta < \min\{2 - q^{(-)}, q^{(+)} - 2\}$.

Step 2. A priori estimates for the Dirichlet problem

$$((1))\qquad \begin{gathered} -\Delta u = \lambda\,[u(x)]^{q(x)-1} + h(x) \quad \text{ for } x \in \Omega\,; \\ u = 0 \text{ on } \partial\Omega\,, \quad \text{ and } u > 0 \text{ in } \Omega\,; \end{gathered}$$

with the **variable exponent** $q : \overline{\Omega} \to \mathbb{R}$
which is assumed to be continuous.

In accordance with Step 1, we work in the set
$\Omega_+ \stackrel{\text{def}}{=} \{x \in \Omega : q(x) > 2\}$.
The well-known identity, for $u, \phi \in W_0^{1,2}(\Omega)$,

$$\int_\Omega (-\Delta u(x)) \cdot \phi(x)\,\mathrm{d}x = \int_\Omega u(x) \cdot (-\Delta \phi(x))\,\mathrm{d}x\,,$$

yields

$$(7)\qquad \begin{aligned} &\lambda \int_\Omega [u(x)]^{q(x)-1} \cdot \phi(x)\,\mathrm{d}x \\ &+ \int_\Omega h(x) \cdot \phi(x)\,\mathrm{d}x = \int_\Omega u(x) \cdot g(x)\,\mathrm{d}x \\ &= \int_{\{x\in\Omega:\, g(x)>0\}} u(x) \cdot g(x)\,\mathrm{d}x \\ &= \int_{\{x\in\Omega:\, q(x)\geq 2+\delta\}} u(x) \cdot g(x)\,\mathrm{d}x\,. \end{aligned}$$

Since $u > 0$ in Ω, the a priori estimates on "a suitable L^p-norm" of u are obtained from the last equation in a way analogous to that developed in the articles by R. D. Nussbaum (1975), H. Brézis and R. E. L. Turner (1977), and D. G. Figueiredo, P.-L. Lions, and R. D. Nussbaum (1982).

We decompose the nonnegative function $u : \Omega \to \mathbb{R}$ as

$$\epsilon \cdot u(x) = \left[(\epsilon \cdot u(x))^{q(x)-1} \cdot \phi(x) \right]^{1/(q(x)-1)} \cdot \phi(x)^{-1/(q(x)-1)},$$

with $\epsilon > 0$ and apply **Young's inequality** to estimate the integral on the right-hand side of eq. (7) to get

$$
\begin{aligned}
\int_\Omega u(x)\cdot g(x)\,\mathrm{d}x &= \int_\Omega \epsilon u(x)\cdot \epsilon^{-1} g(x)\,\mathrm{d}x \\
&\le \int_\Omega \frac{\epsilon^{q(x)-1}}{q(x)-1}\,[u(x)]^{q(x)-1}\,\phi(x)\,\mathrm{d}x \\
&+ \int_\Omega \frac{q(x)-2}{q(x)-1}\cdot \epsilon^{-(q(x)-1)/(q(x)-2)} \\
&\qquad \times \left[\phi(x)^{-1}\cdot g(x)^{q(x)-1}\right]^{1/(q(x)-2)}\,\mathrm{d}x\,.
\end{aligned}
\tag{8}
$$

Substituting

$$\varepsilon = \epsilon^{q(x)-1}/(q(x)-1) > 0$$

we get

$$\epsilon \equiv \epsilon(x) = [\varepsilon(q(x)-1)]^{1/(q(x)-1)}\ .$$

Consequently, inequality (8) becomes

$$
\begin{aligned}
&\int_\Omega u(x)\cdot g(x)\,\mathrm{d}x \\
&\le \varepsilon\cdot\int_\Omega [u(x)]^{q(x)-1}\cdot\phi(x)\,\mathrm{d}x \\
&+\int_\Omega \frac{q(x)-2}{q(x)-1}\cdot\varepsilon^{-1/(q(x)-2)}\,(q(x)-1)^{-1/(q(x)-2)} \\
&\qquad\times\left[\phi(x)^{-1/(q(x)-1)}\cdot g(x)\right]^{(q(x)-1)/(q(x)-2)}\,\mathrm{d}x \\
&=\varepsilon\cdot\int_\Omega [u(x)]^{q(x)-1}\cdot\phi(x)\,\mathrm{d}x \\
&+\int_\Omega \varepsilon^{-1/(q(x)-2)}\cdot\frac{q(x)-2}{(q(x)-1)^{1+\frac{1}{q(x)-2}}} \\
&\qquad\times\left[\phi(x)^{-1}\cdot g(x)^{q(x)-1}\right]^{1/(q(x)-2)}\,\mathrm{d}x \\
&=\varepsilon\cdot\int_\Omega [u(x)]^{q(x)-1}\cdot\phi(x)\,\mathrm{d}x \\
&+\int_\Omega \varepsilon^{-1/(q(x)-2)}\cdot(q(x)-2)\,(q(x)-1)^{-\frac{(q(x)-1)}{(q(x)-2)}} \\
&\qquad\times\left[\phi(x)^{-1}\cdot g(x)^{q(x)-1}\right]^{1/(q(x)-2)}\,\mathrm{d}x\,.
\end{aligned}
\tag{9}
$$

We estimate the left-hand side of eq. (7) by ineq. (9) as follows, provided $0<\varepsilon<\lambda$.

We recall that, for all $x \in \Omega$ we have

$$g(x) > 0 \implies q(x) \geq 2 + \delta\,.$$

$$(10) \qquad \begin{aligned} &(\lambda - \varepsilon) \int_\Omega [u(x)]^{q(x)-1} \cdot \phi(x)\,\mathrm{d}x \\ &+ \int_\Omega f(x) \cdot \phi(x)\,\mathrm{d}x \leq C_\varepsilon < \infty \qquad \text{where} \\ &C_\varepsilon \stackrel{\text{def}}{=} \int_\Omega \varepsilon^{-1/(q(x)-2)} \cdot \frac{q(x)-2}{(q(x)-1)^{\frac{(q(x)-1)}{(q(x)-2)}}} \\ &\qquad \times \left[\phi(x)^{-1} \cdot g(x)^{q(x)-1}\right]^{1/(q(x)-2)}\,\mathrm{d}x\,. \end{aligned}$$

This inequality clearly imposes an upper bound on the solution u for any fixed value of $\lambda \geq \lambda_0 \equiv \text{const} > 0$.

It, ineq. (10), remains valid even if u is only a nonnegative subsolution, i.e., if

$$
\begin{aligned}
&-\Delta u \geq \lambda\,[u(x)]^{q(x)-1} + h(x) \qquad \text{for } x \in \Omega\,; \\
&u = 0 \ \text{ on } \ \partial\Omega\,, \qquad \text{and } u > 0 \text{ in } \Omega\,.
\end{aligned}
\tag{11}
$$

Theorem 1. *Assume that the* **variable exponent** $q : \overline{\Omega} \to \mathbb{R}$ *is a continuous function that satisfies hypothesis* **($\mathbf{H}_q$)**

$$
\begin{aligned}
1 < q^{(-)} &\overset{\text{def}}{=} \min_{\overline{\Omega}} q(x) < p(x) \equiv p = 2 \\
&< q^{(+)} \overset{\text{def}}{=} \max_{\overline{\Omega}} q(x) < \infty\,.
\end{aligned}
$$

In addition, let $0 < \varepsilon < \lambda < \infty$.

Assume that $u \in C_0^1(\overline{\Omega}) = C_0^1(\overline{\Omega}) \cap W_0^{1,2}(\Omega)$ *is any non-negative subsolution to the Dirichlet problem* (1), *such that* u *verifies the* **Hopf maximum principle***:*

(HMP) $u_0 > 0$ *in* Ω *and* $\frac{\partial u_0}{\partial \boldsymbol{\nu}}(x) < 0$ *on* $\partial\Omega$*,*

together with $u \geq u_\varepsilon$ *in* Ω*, where* $u_\varepsilon \in C_0^1(\overline{\Omega})$ *is the unique positive solution constructed in* **Proposition 1** *for the concave nonlinearity* (2) *and* (3)*.*

Then this subsolution, u*, obeys the* **a priori estimate** (10) *above.*

$$\|u_\lambda\|_{C^1(\bar{\Omega})} \leq C \equiv \text{const} \equiv C(\alpha, \beta) < \infty\,.$$

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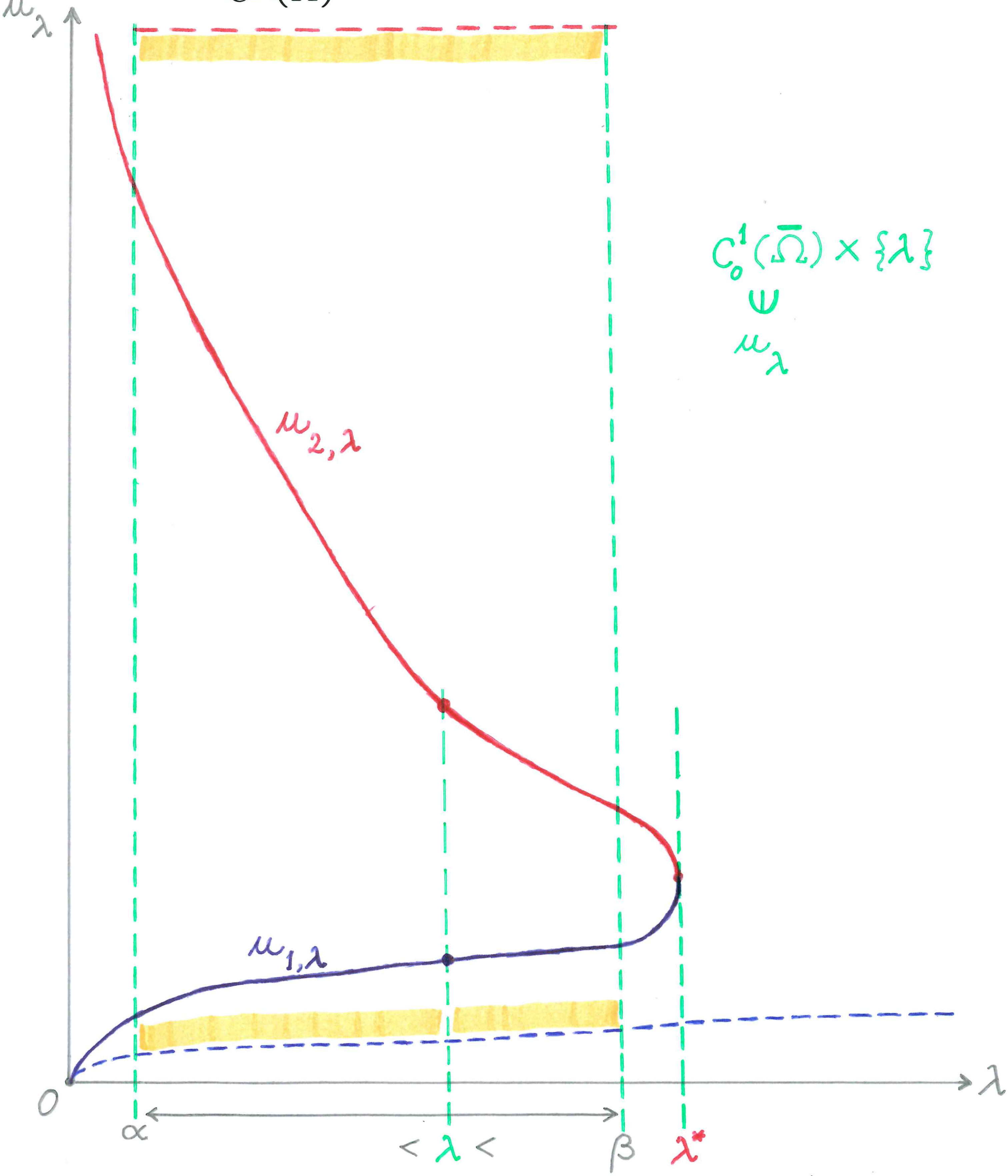

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