Mathematics and Practice

Historical, philosophical & educational perspectives

Interdisciplinary Symposium - Abstracts

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University of Rostock

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Venue

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Abstracts – Keynotes

Verifying informal proofs in mathematical practice: Implications for mathematics education

Gila Hanna

Concepts of mathematical proof have varied widely from one period and place to another, and standards of rigor in particular have changed markedly over time (Grabiner, 1986; Kleiner, 1991). Today the universally accepted definition is that a proof is a finite sequence of propositions, each of which is an axiom or follows from preceding propositions by the rules of logical inference. But in mathematical practice, the great majority of proofs offered and accepted are informal proofs, ones that do not conform to this definition. Although they represent rigorous arguments, they typically consist of a mixture of natural language and formulae. These informal proofs may omit routine logical inferences, but they often have the advantage of conveying greater insight and understanding. Not all practicing mathematicians, however, are content with such informal proofs and "comfortable that the idea works" (Thurston, 1994, p. 168). In response to a growing concern for the correctness of informal proofs, there is now a trend in mathematical practice towards verifying informal proofs through their formalization (Avigad and Harrison, 2014; Voevodsky, 2014; Ganesalingam and Gowers, 2016). The advent of digital proof checkers such as Automated Theorem Provers (ATPs) and Interactive Theorem Provers (ITPs), along with their growing use as tools in mathematics practice, has facilitated this trend. The paper will describe the development of these tools over time, showing how proof checking through formalization has progressed from quite basic methods to a sophisticated incorporation of known patterns of mathematical reasoning. It will also discuss implications for mathematics education of the trend to automated verification.

Empirical Philosophy of Mathematics: Methodological and disciplinary reflections

Benedikt Löwe

In this talk, we shall look at "empirical philosophy of mathematics" and reflect on disciplinary boundaries, whether our research area is a discipline or not, whether it is part of philosophy or not, and what our methodology is or should be. We relate our reflections to the general discussion about "experimental philosophy" and its links with our research community.

Abstracts – Sections

Braids in Italy in the 20th century: material practices of strings

Michael Friedman

The braid group was officially considered as a mathematical object to be investigated in 1925; indeed, starting this year, a flourishing of the mathematical investigation of braids took place. This was prompted mainly due to Emil Artin's 1926 paper "Theorie der Zöpfe", which aimed – at least taking into consideration Artin's official statements – at an algebraic treatment of this group.

However, between the 1930s-1950s, another approach to the investigation of braids was attempted, emphasizing how material three-dimensional models and two-dimensional diagrams are essential for the mathematical investigation. Oscar Chisini and several of his students (Modesto Dedò and Cesarina Tibiletti, for example) investigated braids within the context of algebraic geometry and complex curves. The material models Chisini and his students used were made from, for example, thick threads. But although Chisini emphasized that these models were meant to concretely visualize braids, helping the "visual intuition", they were slowly marginalized from the research papers, and were hardly mentioned, replaced mainly by diagrams. Taking the research of braids in Italy as a case study, I would like to examine if three-dimensional models of braids offered at that time a different kind of reasoning when compared to two-dimensional diagrams. Were the visual arguments dependent essentially on the two-dimensional diagrams or on the three-dimensional models? Why was the tradition of three-dimensional models of braids disappearing in Italy? And what were the differences between the mathematical discoveries, which were prompted uniquely by the material models of braids compared to the discoveries prompted by the diagrams of them?

Instructional situations and their role in describing classroom mathematical practice

Patricio Herbst

How should mathematical practice inform teaching? Descriptions of the mathematical practices of mathematicians have a natural claim on informing mathematics instruction. But what may such informing entail? Questions as to the warrant for the right to transport mathematical practices into instruction have been raised in the past. Chazan (1990) questioned the notion that the presence of certain practices in mathematical research is enough of a warrant to claim that all students should experience such practices.

And yet, the study of mathematical practice can play an essential role organizing how the discipline of mathematics exercises its influence on instruction, how it presses teachers to recognize an obligation to the discipline (Herbst & Chazan, 2012; Chazan, Herbst, & Clark, 2016). This pressure to recognize an obligation to the discipline creates opportunities for the epistemological ties between the practice of mathematics instruction and those of the discipline to continue to evolve, possibly making more room for the representation of mathematical practices or for better representation of some mathematical practices in instruction. Expectations for such evolving presence of mathematical practices in instruction need to be tempered by understanding that the environments in which mathematics instruction exists are not only affected by an obligation to the discipline but also to other stakeholders.

Thus, the meaning of informing needs to be seen as relative to the positionality of whoever makes such a prescriptive claim. From the position of the discipline of mathematics, it is important agents of the discipline press to shape classroom practice into something close to the practices of mathematicians, while at the same time its advocates acknowledge that those intentions need to be negotiated with those of other stakeholders. From the position of an observer of the system of mathematics instruction and all its stakeholders, it is important for the observer to understand the epistemological relationships between the practices of mathematicians that are advocated for inclusion in instruction and the sui-generis practices that obtain in classrooms, possibly as a result of the complex process of transposing (Chevallard, 1991) the former into these different environments. Instructional situation is a useful conceptual tool to describe the units of work that organize classroom mathematical practice. Instructional situations in a course of studies, such as constructing figures and doing proofs, illustrate how canonical mathematical practices have been transposed into mathematics instruction to enable students to participate in mathematical work. Instructional situations can be modeled as systems of norms that describe the division of labor over knowledge; such decomposition helps compare how practices such as constructing and proving differ between classroom mathematics and mathematical research. Inasmuch as these instructional situations refer to stable and recurrent practices, they can be seen as satisficing the various obligations that subtend instruction and the work of teaching. Thus, instructional situations are useful examples to describe what may successfully transposed mathematical practices look like in classrooms.

Mathematical practice at school: Forms of proof in school mathematics considered as ‚diagrammatic reasoning‘

Leander Kempen

Charles S. Peirce considers mathematical practice as ‚diagrammatic reasoning' (e.g., Hofmann, 2005). This semiotic theory can be used to describe the proving process in mathematics (Kempen, 2019). I will use this view on mathematical proof to discuss forms of proof and proving in the classroom. Making use of Stylianides (2007) definition of ‚proof', dagrammatic reasoning can be used to advocate suitable forms of mathematical proof for school mathematics.

Generation of mathematical knowledge through heuristic refutation

Kotaro Komatsu & Keith Jones

Mathematical philosopher Imre Lakatos described one aspect of mathematical practice where mathematics develops through mathematical activity involving proofs and refutations. In our PRinDGE project, we have aimed to introduce this activity into school mathematics so that students can experience authentic mathematical practice. To date, we have shown how tasks can be designed to engage students in mathematical activity consisting of conjecturing, proving, and refuting (Komatsu, 2017; Komatsu & Jones, in press). We are currently adding another dimension, the generation of mathematical knowledge (such as mathematical definitions and theorems), to this mathematical activity. This is consistent with Lakatos's research in which he argued that some mathematical definitions (e.g., polyhedra and uniform convergence) were proposed during the process of dealing with counterexamples. In this presentation, we report on an intervention study implemented in a lower secondary school in Japan (students aged 14–15). In the implemented lessons, tasks designed using specific principles, and the teacher's roles, supported the students in finding and proving the inscribed quadrilateral theorem through addressing counterexamples they discovered.

The Material Reasoning of Folding Paper

Colin Rittberg

Fold a piece of paper flat onto itself and open it up again. The resulting crease is a straight line. Fold again such that the crease is folded onto itself. The resulting crease is perpendicular to the first. Thus, by folding paper we can construct lines in a controlled fashion; paper-folding allows *for doing geometry. In this talk I present paper-folding as a material reasoning practice with roots in thoughts about mathematics education and discuss the epistemic force of folding-proofs.*

This is joint work with Michael Friedman.

Using prompts to scaffold mathematical argumentation

Daniel Sommerhoff & Stefan Ufer

Successfully handling mathematical argumentations and proofs obviously requires content knowledge, for example regarding mathematical definitions or theorems. However, such content knowledge is not sufficient. Other knowledge facets and skills are required as resources underlying mathematical argumentation and proof skills, such as multiple domain-specific or domain-general strategies to approach (mathematical) problems (e.g., Chinnappan, Ekanayake, & Brown, 2012; Sommerhoff, Ufer, & Kollar, submitted; Ufer, Heinze, & Reiss, 2008). Moreover, knowledge about what constitutes evidence can be deemed as important to handle mathematical argumentations and proofs proficiently: This not only includes knowledge about the nature and functions of mathematical proofs (de Villiers, 1990; Hanna, 1990; Heinze & Reiss, 2003), but also knowledge about argumentation and proof as social practices, corresponding local socio-mathematical norms, and acceptance criteria for mathematical proofs (Hemmi, 2006; Sommerhoff & Ufer, submitted; Yackel & Cobb, 1996).

Empirical research has underlined repeatedly that students on all educational levels have difficulties handling mathematical proofs (Healy & Hoyles, 2000; Weber, 2003), and that problems can be traced back to different resources underlying mathematical argumentation and proof skills (e.g., Chinnappan et al., 2012; Sommerhoff et al., submitted; Weber, 2001). Accordingly, one promising path for mathematics educators at school and university to promote proof-related skills is to support the acquisition and development of these resources in each student and to foster their successful implementation, for example during proof construction and proof validation.

One approach to support students in developing complex skills such as proving is scaffolding (Wood, Bruner, & Ross, 1976). Scaffolding describes an instructional strategy providing temporary support for students, that is adapted to their individual learning status. It relies on the zone of proximal development (Vygotsky, 1980) and supports students to successfully master tasks, which are regularly slightly out of reach of their current skills. Scaffolding is particularly well suited for channeling and focusing students on individual aspects of tasks, for example on implementing one of the resources needed at a specific moment, as well as to model how to do so (Pea, 2004). A particularly well researched scaffolding approach is the use of prompts, which have been used over different contexts and disciplines (e.g., Bannert, 2009; Kollar et al., 2014). In combination with different fading strategies, scaffolding by prompts is considered as a light-weight form of support that can enable effective, long-term learning (see further Reiser & Tabak, 2014).

The talk focuses on different approaches to use prompts as scaffolds during the construction and validation of mathematical argumentations and proofs, in order to develop the various resources underlying students' argumentation and proof skills, as well as to foster their implementation during mathematical argumentation practices. A particular focus will be given on prompts regarding the acceptance criteria for proofs and the various functions of proof.

Conviction, empirical evidence, and proof: An empirical study of mathematical practice

Keith Weber, Juan Pablo Mejia-Ramos, Tyler Volpe

The purpose of this presentation is to explore how mathematicians obtain psychological certainty in mathematical facts. In the first part of the talk, we present two points of view: Proofs Provide Certainty which indicates that mathematicians are certain of a claim exactly when they validate a proof of that claim and Confluence Can Provide Certainty which asserts that if a mathematician gains certainty, she usually uses a combination of deductive, empirical, and testimonial evidence to obtain it. We illustrate how the Proofs Provide Certainty perspective has had a *large influence in mathematics education. In particular, many mathematics education scholars regard students as epistemologically naïve if, after seeing a proof of that claim, the students retain some doubt that the claim is true or continue to seek empirical evidence to support or refute that claim.*

In the second part of the talk, we present an empirical study challenging the Proofs Provide Certainty perspective. In a task-based interview with 16 mathematicians, we examined how various types of evidence in support of a specific mathematical claim increased mathematicians' confidence in that claim. In an open-ended interview, we explored how these mathematicians coordinate multiple forms of evidence in deciding what to believe. The mathematicians' responses in both the task-based interviews and the open-ended interviews were inconsistent with the Proofs Provide Certainty position. Most mathematicians said they would not be certain of a claim after reading a proof of that claim and they would continue seeking further evidence of that claim after reading that proof. Consequently, mathematicians consciously engaged in actions that mathematics educators have negatively evaluated students for doing.

In the third part of the talk, we note that although the Confluence Can Provide Certainty currently lacks the explicitness to make firm predictions on how mathematicians will behave in specific circumstance, the data from our study is broadly consistent with that position. We discuss some open philosophical research questions that the Confluence Can Provide Certainty position raises that have received limited attention from both philosophers and mathematics educators. In particular, we consider the issue of what constitutes a good mathematical (or mathematics classroom) conjecture? When do mathematicians (and when should students) have doubt that a proof is correct and what types of actions should they take to resolve these doubts?

List of Speakers

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