Stochastik II
für Lehrämter und Physiker

PD Dr. Reinhard Mahnke

Lehrveranstaltung Nr. 11572
(2 SWS V + 2 SWS Ü)

Dienstag 7.30 bis 9.00 Uhr, Seminarraum R306
Mittwoch 7.30 bis 9.00 Uhr, Seminarraum R306
Institut für Physik, Universitätsplatz 1
Sommersemester 2008

This is a joint lecture on *Stochastic Processes* from the mathematical (Prof. Dr. Friedrich Liese, LV 11334) as well as physical (PD Reinhard Mahnke, LV 11572, see below) points of view.

About the contents:

1. **Basic concepts of deterministic dynamics**
   (02.04.2008, R. Mahnke)
   Defining dynamical system by n-dimensional state space (or phase space) and a mapping including set of control parameters as causal relationship. Time-discrete dynamics as iterated maps; time-continuous dynamics as coupled set of ordinary differential equations, always together with initial conditions. Dynamical system has unique solution called trajectory. Topological consequences like expanding/contracting or conservative dynamics are investigated within the so-called *qualitative theory of dynamical system*. Important classes are: Gradient system, canonical-dissipative system. Stability of regular or irregular (chaotic) motion measured by Ljapunov exponents.

2. **Qualitative theory of dynamical systems** (09.04.2008, R. Mahnke)
   Recalling the definition of a dynamical system (x, T) consisting of an n–dimensional state space x and a nonlinear discrete or continuous mapping T together with initial conditions. Examples are iterations, Newtonian dynamics, Hamiltonian canonical equations, overdamped motion in a potential field, etc. Key words in the *qualitative theory* (in distinction to the exact trajectory solution) are: stationary or steady state solutions (fixed point, periodic orbit), separatrix, saddle point, region of attraction, linear stability analysis (with respect to small changes), Ljapunov exponents, chaotic attractor, ...

3. **Examples of low–dimensional dynamical systems** (16.04.2008, R. Mahnke)
   The following home works should be performed (till 21.04.2008):

   (a) Mathematical pendulum ([1], Chap. 2)

   $$\frac{d}{dt}\alpha = \frac{p_\alpha}{ml^2} \quad (1)$$

   $$\frac{d}{dt}p_\alpha = -mgl\sin\alpha \quad (2)$$

   →Kerstin Witte

   (b) Chairoplane ([1], Chap. 3)

   $$\frac{d}{dt}\alpha = \frac{p_\alpha}{ml^2} \quad (3)$$

   $$\frac{d}{dt}p_\alpha = ml^2\omega_0^2[-\sin\alpha + \Omega(A + \sin\alpha)\cos\alpha] \quad (4)$$

   →Karsten Dittrich

   (c) Van der Pol–oscillator ([1], Chap. 6.1)

   $$\frac{d}{dt}x = x - y - x(x^2 + y^2) \quad (5)$$

   $$\frac{d}{dt}y = x + y - y(x^2 + y^2) \quad (6)$$

   →Siegfried Sobkowiak

   (d) Predator–prey–dynamics, Lotka–Volterra–system ([1], Chap. 6.2)

   $$\frac{d}{dt}x = k_1Ax - k_{12}xy \quad (7)$$

   $$\frac{d}{dt}y = k_{21}xy - k_2y \quad (8)$$

→Sebastian Dittrich

(e) Three species food chain ([1], Chap. 6.7, and [2])

$$\frac{d}{dt}x = R_0 x \left(1 - \frac{x}{K_0}\right) - C_1 F_1(x) y \tag{9}$$

$$\frac{d}{dt}y = F_1(x)y - F_2(y)z - D_1 y \tag{10}$$

$$\frac{d}{dt}z = C_2 F_2(y) z - D_2 z \tag{11}$$

→Hannes Hartmann

(f) Lorenz model ([1], Chap. 6.5)

$$\frac{d}{dt}x = s(y - x) \tag{12}$$

$$\frac{d}{dt}y = rx - y - xz \tag{13}$$

$$\frac{d}{dt}z = xy - bz \tag{14}$$

→Arian–Christoph Pfahl

(g) Logistic mapping ([1], Chap. 8.1)

$$x_{n+1} = r x_n (1 - x_n) \tag{15}$$

→Alko Schurr + Jan Trautmann

(h) Brusselator ([1], Chap. 12.4)

$$R_1 \to X \tag{16}$$

$$R_2 + X \to Y + F_1 \tag{17}$$

$$2X + Y \to 3X \tag{18}$$

$$X \to F \tag{19}$$

→Lennart Forck

(i) Schlögl reaction ([1], Chap. 13)

$$A + 2X \rightleftarrows 3X \tag{20}$$

$$X \rightleftarrows F \tag{21}$$

→Michael Kelbg

(j) Magnitskii's ODE–system ([3], Example 2)

$$\dot{x}_1 = \mu x_1 - \nu x_2 - x_1^2 x_3^2 \tag{22}$$
$$\dot{x}_2 = \nu x_1 + \mu x_2 - x_1 x_2 x_3^2 \tag{23}$$
$$\dot{x}_3 = x_1^2 + x_2^2 + x_3^2 - \sigma x_3 \tag{24}$$

→Johannes Knebel + Falk Töppel

4. **Presentations I** (22.04.2008, R. Mahnke)
The following home works have been presented by students:

   (a) Kerstin Witte: Mathematical pendulum
   Lagrange function, Hamiltonian, equations of motion, initial conditions, conservation of mechanical energy, general trajectory, phase diagram showing four different cases (equilibrium, libration, separatrix, rotation), dynamics on separatrix, fixed points

   (b) Karsten Dittrich: Chairoplane
   Gravitational and centrifugal forces, dimensionless control parameters, potential energy (calculated from total energy), discussion of different cases (monostability, bistability)

   (d) Sebastian Dittrich: Predator–prey–dynamics
   Lotka model, extension to Lotka–Volterra system, two fixed points and their stability analysis (hyperbolic refers to extinction; elliptic means oscillations between predator and prey)

5. **Langevin equation I** (23.04.2008, R. Mahnke)
Adding a white noise term (stochastic part) to the one–dimensional dynamical system (deterministic part), this generates a stochastic differential equation known as Langevin equation. This empirical equation consisting of a drift and a diffusion part together with the initial condition has a formal solution as stochastic trajectory. A special case without drift is called Wiener process or white noise. The properties of the noise are given by mean value and correlation function. The double well potential which corresponds to a cubic deterministic force is of special interest to get the moments of the stochastic process, the correlator as well as the response. An outlook to the Fokker–Planck equation related to the discussed Langevin equation is given.

6. **Presentations II** (29.04.2008, Ch. Liebe)
   The following home works have been presented by students:

   (c) Siegfried Sobkowiak: Van der Pol–oscillator
   Historical background, general behaviour, discussion of two special cases ($\epsilon = 0$ and $z^2 \ll 1$), detailed discussion of transformed version for $\epsilon = 1$ (analytical solution, graphical presentation of solution)

   (e) Hannes Hartmann: Three species food chain
   General discussion of the problem, introduction of the logistical differential equation, short discussion of predator–prey–dynamics, detailed discussion of the three species food chain and presentation of numerical results (Maple) (three examples: eating rate of predators is zero, stable fix point solution, solution with chaotic attractor)

   (j) Johannes Knebel + Falk Töppel: Magnitskii's ODE–system
   Short introduction to Magnitskii's ODE–system, detailed presentation of numerical results (MatLab), with fixed initial values and fixed control parameters except μ, one can show that the general shape of the attractor strongly depends on the parameter μ, changing the initial values can lead to a singularity

   (h) Lennart Forck: Brusselator
   Short historical introduction, derivation and formulation of the kinetic equations, discussion of the equilibrium point and its stability, presentation of numerical results (program written in C) showing repelling and attractive behaviour of the fixed point

7. **Presentations III** (30.04.2008, Ch. Liebe)
   The following home works have been presented by students:

   (f) Arian–Christoph Pfahl: Lorenz–model
   Historical introduction, description of an experiment, analytical discussion of the equations, stationary solutions and bifurcation, fixing two parameters and varying the third one leads to Lorenz attractor, other examples for Lorenz–model

   (i) Michael Kelbg: Schlögl–reaction
   Historical introduction, formulation of the equations, transformation leads to a nonlinear differential equation, interpretation as a force leads to potential, stationary solutions (bistable or monostable depending on the parameter), graphical presentation of the results, connection to Van der Waals gas

(g) Alko Schurr + Jan Trautmann: Logistic mapping
Historical introduction, derivation of a demographic model to predict the development of a population, transformation to decrease number of parameters, changing a parameter defined between zero and four leads to different long time results, deterministic chaos, islands of order, graphical presentation of numerical results

8. **Seminar about home works** (06.05.2008, R. Mahnke)
The following exercises have been discussed:

   (a) Consider a Langevin particle starting at $x(t=0) = x_0$ under the influence of a given drift function $f(x) = -\alpha x - \beta x^3$ as well as a given diffusion as white noise $\xi(t)$ or Wiener process $dW(t)$. Calculate the central tendency as mean or average of that stochastic process.

   (b) Calculate the stationary solution and explain the supercritical bifurcation by changing the control parameters α and β. How does the subcritical bifurcation differ from the supercritical one?

9. **Langevin equation II** (07.05.2008, R. Mahnke)
Extension of a deterministic differential equation by a stochastic force (additive white noise) gives the Langevin equation. Important as equivalent description of stochastic trajectories is the Fokker–Planck equation, which shows the development of the probability density of a stochastic drift–diffusion process. Discussing two different situations depending on the highest order force term x^n: (a) $n = 3$, supercritical bifurcation (2nd order phase transition, double well potential);

$$dx = \left(-\alpha x - \beta x^3\right) dt + \sigma \, dW(t) \qquad (25)$$

(b) $n = 5$, subcritical bifurcation (1st order phase transition with jumps and hysteresis). Investigating the simplest case (without deterministic force) and the simple case (linear force, $n = 1$) in detail including numerical hints for simulation technique.

Study of Langevin dynamics with linear force (monostability) given by

$$\frac{dv(t)}{dt} = -\alpha v(t) + \sigma \xi(t) \quad ; \quad v(t=0) = v_0 \ . \qquad (26)$$

Stochastic trajectory as exact solution is obtained

$$v(t) = v_0 \exp(-\alpha t) + \sigma \exp(\alpha t) \int_0^t \exp(-\alpha s) \, \xi(s) ds \ . \qquad (27)$$

The time dependent first moment is given as

$$\langle v(t) \rangle = v_0 \exp(-\alpha t) \to 0 \quad \text{if} \quad t \to \infty \, . \tag{28}$$

The time dependent second moment is given as

$$\langle v(t)^2 \rangle = v_0^2 \exp(-2\alpha t) + \frac{\sigma^2}{2\alpha} \left(1 - \exp(-2\alpha t)\right) \, . \tag{29}$$

We get for the variance

$$\langle v(t)^2 \rangle - \langle v(t) \rangle^2 = \frac{\sigma^2}{2\alpha} \left(1 - \exp(-2\alpha t)\right) \to \frac{\sigma^2}{2\alpha} \quad \text{if} \quad t \to \infty \, . \tag{30}$$

Fluctuation–dissipation theorem holds in equilibrium

$$\frac{\sigma^2}{2\alpha} = \frac{k_B T}{m} \quad \text{from} \quad \left\langle \frac{m}{2} v^2 \right\rangle = \frac{k_B T}{2} \, . \tag{31}$$

2dim linear extension (coordinate–velocity space) including special case $D = 0$ called Ornstein–Uhlenbeck process

$$dx = v \, dt + \sqrt{2D} \, dW(t) \tag{32}$$

$$dv = -\gamma v \, dt + \sqrt{2B} \, dW(t) \tag{33}$$

2dim extension in coordinate–velocity space: oscillator model with noise

$$dx = v \, dt \tag{34}$$

$$dv = -\omega_0^2 x \, dt - \gamma v \, dt + \sqrt{2B} \, dW(t) \tag{35}$$

10. **Langevin equation III** (13.05.2008, R. Mahnke)
The Brownian gas model as a system of many Brownian particles with interaction is considered. The dynamics of this many–body system is given as a set of Langevin equations. Usually thermal equilibrium is treated where the fluctuation–dissipation theorem (Einstein relation) is valid and the canonical distribution holds. The overdamped limit neglecting momenta is discussed. The concept of active Brownian particles is introduced. Taking into account a velocity–dependent damping function instead of constant friction coefficient in the Langevin equation the Brownian particles are called active ones. Typical models with short–range interaction (named next neighbour) are Morse or Toda ring chains with periodic boundary conditions. These models exhibit new phenomena like clustering effects.

11. **Langevin equation IV** (14.05.2008, R. Mahnke)
    As extension of additive white noise term the Langevin equation with multiplicative noise (state dependent fluctuations) is introduced. Depending on the point of integration the stochastic differential equation is of different type called Ito (left border of integration interval), Stratonovich (middle) or Hänggi–Klimontovich (right). The relationship to the general corresponding Fokker–Planck equation is given as well as the transformation rules for drift and diffusion terms. Finally a special case called geometric Brownian motion in Ito notation is considered in comparison to the arithmetic Brownian development.

12. **Fokker–Planck equation I** (20.05.2008, R. Mahnke)
    One of the fundamental dynamical expressions for Markovian processes is the Fokker–Planck equation (FPE) in its forward and backward notation. Discussing the multi–dimensional forward FPE written as continuity equation and the corresponding backward FPE together with the same delta–like initial condition in both cases. Taking into account borders we consider the one–dimensional bounded drift–diffusion problem in a finite interval with a reflecting (left) and an absorbing (right) wall and state that both Fokker–Planck dynamics with the corresponding boundary conditions are equivalent and give the same result.

13. **Fokker–Planck equation II** (21.05.2008, R. Mahnke)
    We consider the typical one–dimensional exit problem of a Brownian particle (drift–diffusion dynamics) from a bounded domain, whose boundary is usually reflecting except for an absorbing window. The escape problem and its solution is given by the outflow function as first passage time probability density. The first moment is called mean first passage time (MFPT) whereas the inverse is known as escape or breakdown rate. The MFPT tells us how long does it take in the average to move from the initial value inside the interval to the right open boundary taking into account the left reflecting wall. The result for the dynamics with constant drift as well as constant diffusion is shown graphically.

14. **Performing Projects I** (27.05.2008, all students)
    Given tasks to solve as project at home:

    (a) **Random numbers** → Mathias Richter
        Although a computer is a deterministic machine, it is possible to create good random numbers with it. Describe at least one algorithm to create equally distributed random numbers with a

computer. Implement this algorithm in a language of your choice. Describe at least two algorithms to create standard normal distributed random numbers out of equally distributed random numbers. Implement these algorithms in a language of your choice and compare the quality of the produced random numbers and the efficiency of the algorithms.

(b) **Ornstein–Uhlenbeck process** → Arian–Christoph Pfahl
The Ornstein-Uhlenbeck process is defined in the coordinate–velocity space by the following differential equations

$$dx = v\,dt \tag{36}$$

$$dv = -\gamma v\,dt + \sqrt{2B}\,dW(t) \tag{37}$$

with $x(t=0) = x_0$ and $v(t=0) = v_0$.
The same system can be described by a Fokker-Planck equation

$$\frac{\partial}{\partial t} p(x,v,t) = -\frac{\partial}{\partial x}[vp(x,v,t)] + \frac{\partial}{\partial v}[\gamma v p(x,v,t)] + \frac{\partial^2}{\partial v^2}[Bp(x,v,t)] \tag{38}$$

with initial condition

$$p(x,v,t=0) = \delta_{x-x_0}\delta_{v-v_0} \tag{39}$$

Solve this system temporally for stochastic trajectories as well as for probability density (use Fourier transformation method) and discuss the overdamped limit as approximative result.

(c) **Geometric Brownian motion** → Falk Töppel
Start with the following stochastic differential equation in Ito notation

$$dx(t) = a\,x(t)dt + b\,x(t)dW(t) \quad ; \quad x(t=0) = x_0 > 0 \tag{40}$$

and investigate as much as possible (list of proposals: solution as stochastic trajectory for different cases including critical situation $a = b^2/2$, moments, variance, transformation to other notations like Stratonovich and Hänggi–Klimontovich type, probability density as log–normal distribution, relationship to Black–Scholes theory and diffusion equation). Hint: see Ref. [5].

(d) **Brownian motion in monostable and bistable potential →** Johannes Knebel

Consider a phase transition model given by the following stochastic Langevin equation

$$\frac{dx(t)}{dt} = -\alpha\, x(t) - \beta\, x(t)^3 + \sigma\, \xi(t) \qquad ; \qquad x(t=0) = x_0 \quad (41)$$

setting $\beta \geq 0$ and turning the rate α from positive (monostability) over zero (critical case) to negative values (bistability).

Due to cubic nonlinearity ($\beta > 0$) a complete analytic solution $x(t)$ is probably impossible. Repeat the calculations to get the first moment known as noisefree solution

$$\langle x(t)\rangle = \begin{cases} x_0\, e^{-\alpha t} & : \beta = 0\,, \\ \pm\left(x_0^{-2} + 2\beta t\right)^{-1/2} & : \beta > 0\,, \alpha = 0\,, \\ \pm\sqrt{(-\alpha/\beta)\left[1 - \left(1 + x_0^{-2}\alpha/\beta\right) e^{2\alpha t}\right]^{-1}} & : \beta > 0\,, \alpha \neq 0\,. \end{cases} \quad (42)$$

and show the merging of one solution into the other when the control parameter α is changing.

The main task is to get the mean square displacement known as second moment $\langle x(t)^2\rangle$ (or variance $\langle x(t)^2\rangle - \langle x(t)\rangle^2$) as solution of a better mean field equation as we did.

Explain the relaxation into equilibrium ($\alpha > 0$) and into non-equilibrium ($\alpha < 0$) with respect to mean value and variance and calculate the fluctuation–dissipation ratio (follow MECO33 contribution by Malte Henkel et al. [6]).

(e) **The drunken sailor:**
Discrete random walk in one dimension → Alko Schurr
Study the stochastic motion by discrete probabilistic jumps on an (asymmetrically) Galton board. Start with the historical background at the life time of *Sir Francis Galton*. Explain the binomial distribution.

Entwickeln Sie die diskrete Beschreibung der Zufallsbewegung einer Kugel, die durch ein Galton–Brett fällt. Die elementaren Hüpfwahrscheinlickeiten p und $q = 1-p$ seien durch die Geometrie des Galtonschen Glückspielautomaten gegeben. Betrachen Sie auch die beiden Spezialfälle $p = q = 1/2$ (Symmetrie, reine Diffusion)und $p = 0$ bzw. $p = 1$ (totale Asymmetrie, reine Drift).

Lassen Sie sowohl die Hüpfzeit τ als auch die Sprungweite a gegen Null gehen. Führen Sie bei diesem Grenzprozess zwei neue endliche Parameter D (Diffusionskoeffizient) und v (Driftkoeffizient) ein.

Berechnen Sie in dieser Kontinuumsgrenze den Drift–Diffusions–Prozess als Lösung einer partiellen Differentialgleichung (Fokker–Planck–Gleichung).

See H. Haken: Synergetik, Springer–Verlag, Berlin, div. Auflagen ab 1977 (or [7]) and J. Vollmer: Chaos, spatial extension, transport, and non-equilibrium thermodynamics, Physics Reports 372, Dec. 2002, pp. 131–267, see: Chap 1–3 (pp. 131–163).

(f) **A drunken sailor close to quay** → Hannes Hartmann
In this task we shall continue the discussion of the problem of random walk in one dimension but with certain restrictions on the motion of the particle introduced by the presence of reflecting or absorbing walls. Consider the influence of the boundaries in detail, especially by the perfectly absorbing barrier.

(g) **Brownian motion:**
Continuous random walk in one dimension → Jan Trautmann
The motion named after *Robert Brown* shows the stochastic displacement of a particle by one–dimensional diffusion.

Study Einstein's concept of Brownian motion to derive the well–known diffusion equation by reading the original paper *Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen* in Annalen der Physik 1905, pp. 549–560.

The solution of the diffusion equation is known as Gaussian distribution. Show its profile and discuss the properties, especially the moments.

(h) **Stochastic oscillator with/without friction** → Kerstin Witte
Consider the oscillator model with noise given by

$$dx = v\,dt \tag{43}$$

$$dv = -\omega_0^2 x\,dt - \gamma v\,dt + \sqrt{2B}\,dW(t) \tag{44}$$

and investigate two cases: with ($\gamma > 0$) (What is tribology?) and without ($\gamma = 0$) friction.

(i) **Stochastic van der Pol oscillator** → Siegfried Sobkowiak
Based on the deterministic van der Pol oscillator showing limit cycle behaviour please add stochasticity now. Explain the random system and its development. Does the monography by Guckenheimer & Holmes help you again?

(j) **Stochastic brusselator** → Lennart Fork
Follow section 12.4 in [1] and derive the master equation of the stochastic brusselator. Make numerical realisations.

(k) **Stochastic Schlögl reaction** → Michael Kelbg
Describe the Schlögl reaction from the stochastic point of view. Follow chapter 13 from [1] or similar publications.

(l) **Ehrenfest model of diffusion between two boxes** → Karsten Dittrich
Das Physikerehepaar Paul und Tatiana Ehrenfest formulierte 1907 ein einfaches Modell für die Diffusion von N Molekülen zwischen zwei miteinander durchlässig verbundenen Behältern. Siehe: *Über zwei bekannte Einwände gegen das Boltzmannsche H–Theorem*, Phys. Zeitschr. 1907, 8. Jahrgang, S. 311–314

Die stochastischen Beschreibung der Diffusion in einem Zwei–Boxen–Modell ist bei einem diskreten Zustandsraum (Teilchenzahl n) und kontinuierlicher Zeit durch folgende lineare Master–Gleichung gegeben. Sie lautet

$$\frac{\partial}{\partial t}p(n,t) = d_{21}(N-(n-1))P(n-1,t) + d_{12}(n+1)P(n+1,t) - \left[d_{21}(N-n) + d_{12}n\right]P(n,t)\,. \tag{45}$$

Als Anfangsbedingungen setzen wir

$$P(n, t=0) = \delta_{n-n_0}\,. \tag{46}$$

Diese obige Master–Gleichung (45) ist zeitabhängig zu lösen.

Die analytische Lösung ist bekannt. Vergleichen Sie das Ergebnis mit der determinischen Lösung (Mittelwerte) und visualisieren Sie die Verteilung $P(n,t)$.

Schreiben Sie ein Computerprogramm zur Lösung der stochastischen Master–Gleichung des Zwei–Boxen–Diffusionsprozesses. Generieren Sie stochastische Trajektorien und bilden Sie für ausgewählte Zeitpunkte die entsprechenden Häufigkeitsverteilungen. Vergleichen Sie die numerischen Resultate in Abhängigkeit von der Ensemblegröße mit der analytischen Lösung.

(m) **Cluster decay versus radioactive decay** → Sebastian Dittrich
Take a one–step master equation with detachment terms only. Investigate this stochastic dissolution process, starting from a given (large) size $n(t=0) = n_0 > 0$, within two different models: (a) car

cluster decay with constant transition rate $1/\tau$ and (b) molecular or atomic decay process with linear transition rate αn. Define your system as open or closed depending on the property of the transition rate at the border $n = 0$.

15. **Performing Projects II** (28.05.2008, all students)
    Home work

16. **Performing Projects III** (03.06.2008, all students + R. Mahnke)
    Home work and tutorial work for preparation of presentations.

17. **Stochastic Markovian process** (04.06.2008, R. Mahnke)
    A stochastic process describes the temporal evolution of random events by probability distributions. A stochastic trajectory as a time series (sequence of states and times) is called a realisation. After introduction of joint probability densities (jpd) and conditional probability densities (cpd) and their relationship the two time moments correlated Markov process has been discussed. In contract to factorisation of temporally uncorrelated processes the Markovian dynamics is given by the *Chapman–Kolmogorov integral equation* which can be written in the short time limit as differential equation named *master equation*.

18. **Presentation of Project Works I** (10.06.2008, R. Mahnke)

    ⇒ Matthias Florian: Three–level Markov process
    Starting with an overview about Markovian processes the master equation in continuous as well as in discrete formulation is presented. The three–level system described by the discrete case of master equation is discussed where the initial probabilities are given and all six transition rates are taken as constants. The solution is a typical eigenvalue problem with one zero eigenvalue corresponding to the stationary distribution. Specific problems like detailed balance (all transition rates are equal) or stationary flux on a ring are discussed in detail. Final remarks concern applications and recent investigations (laser description by master equation).

    ⇒ Mathias Winkel: Simulation techniques: Generation of stochastic trajectories & numerical solution of master equation
    Using Euler discretization technique the numerical solution of a stochastic differential equation of Ito type is discussed. Special cases like Wiener process (white noise) and geometrical Brownian motion are presented by generation of stochastic trajectories as well as probability densities. The numerical procedure of solving the stochastic master

equation is explained by using different methods (fixed and variable time step). Simulation results are shown for three models (jam dissolution, cluster decay, traffic flow) and compared in part with analytical solutions.

⇒ Kerstin Witte: Stochastic oscillator
The investigations of the noisy harmonic oscillator with friction are based on the following set of equations of motion

$$\begin{pmatrix} dx(t) \\ dv(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -\beta \end{pmatrix} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW(t) \tag{47}$$

together with given initial conditions $x(t=0) = x_0 \,; v(t=0) = v_0$. The mean value (homogeneous solution, without fluctuations) tends to zero (if damping $\beta > 0$). The exact result $x(t)$ can be calculated by variation of constants as stochastic trajectory. The second moment (or variance) in velocity space is finite known from free Brownian motion. The correlation matrix (or covariance) reads in the long–time limit

$$COV = \frac{\sigma^2}{2\beta} \begin{pmatrix} 1/\omega_0^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{48}$$

19. **Presentation of Project Works II** (11.06.2008, R. Mahnke)

⇒ Falk Töppel: Geometric Brownian motion
The geometric Brownian motion is described by a stochastic differential equation. Two different interpretations are considered: Ito notation and Stratonovich calculus. The solution called stochastic trajectory is given and shown in figures using different control parameters. The solution enables to calculate moments and variance.

⇒ Lennart Fork: Stochastic brusselator
Based on the deterministic model of the Brusselator the stochastic version is developed. The master equation with well–explained transition rates is used to get stochastic realisations. Illustrations in the phase space are shown.

⇒ Siegfried Sobkowiak: Stochastic van der Pol oscillator
In the beginning the Van der Pol oscillator without noise is analysed. The deterministic behaviour shows a limit cycle which is presented in phase space. The general stochastic model with two noise sources is much more complicated. Taking into account one diffusion term only the Fokker–Planck equation is discussed and solved in the stationary case. The long–time solutions is written with the help of a double well potential.

20. **Presentation of Project Works III** (17.06.2008, R. Mahnke)

    ⇒ Hannes Hartmann: A drunken sailor close to quay
    The symmetric random walk of a sailor on a line is studied taking into account a given absorbing border called quay. At first the discrete case is analysed analytically and by simulations. Finally the transition from a walk with discrete steps to a space–time continuum random walk is performed.

    Brownian motion: Continuous random walk in one dimension
    given by Reinhard Mahnke

    Ornstein–Uhlenbeck process
    given by Christof Liebe

21. **Presentation of Project Works IV** (18.06.2008, R. Mahnke)

    ⇒ Karsten Dittrich: Ehrenfest model of diffusion between two boxes
    The diffusion model by Ehrenfest uses two boxes with particle jumps between them. The one–step master equation with simple linear transition rates is written. The general time dependent solution taking into account given initial condition is known but difficult to get. The method using generating functions is helpful, at least to get the stationary solution.

    ⇒ Mathias Richter: Random numbers
    Firstly the terms randomness and random number are defined. The generation of so–called pseudo–random numbers is explained. Different methods of creating random numbers are compared. After implementation in C++ the properties of generators are analysed.

    ⇒ Sebastian Dittrich: Cluster decay versus radioactive decay
    Based on a general one–step master equation two different decay processes are considered in detail. The vehicular cluster decay (jam dissolution) is described by a constant detachment rate. The cluster shrinks linearly. The radioactive decay process uses a linear rate, therefore the cluster becomes smaller exponentially (mean value).

22. **Presentation of Project Works V** (24.06.2008, R. Mahnke)

    ⇒ Johannes Knebel: Brownian motion in monostable
    and bistable potential
    Langevin equations are discussed. The cases without force as well as with linear force (quadratic potential) are well known. The situation with cubic force (monostable and bistable potential) is much more complicated.

⇒ Michael Kelbg: Stochastic Schlögl reaction
Stochastic description of Schlögl reaction by one–step master equation. Transition rates are constructed based on chemical kinetics. Simulation results show bistability.

The drunken sailor: Discrete random walk in one dimension
given by Alko Schurr
Explaining Galton board by showing random events empirically.

23. **Randomness in Traffic Flow** (25.06.2008, R. Mahnke)
Celebrating 75 years of the very first paper by Greenshield on speed–flow curves, the so–called *Fundamental Diagram* as steady state relationship of flux over density has its 75th anniversary in July 2008. The empirical traffic flow theory is described as phase transition between free flow and congested traffic. The concept of nucleation on roads is explained by forming of vehicular clusters in analogy to supersaturated vapours. The cluster size as number of bounded vehicles is introduced as stochastic variable. The breakdown probability density has to be calculated out of the given stochastic dynamics.

24. **Cluster Formation on Roads** (01.07.2008, R. Mahnke)
Having the first passage time problem in mind the car cluster formation on roads is discussed as a one–step process (only one car is reaching or leaving the jam). Using well–explained ansatz for the transition rates the master equation is formulated. Defining a particular size of congestion (escape value) and the initial situation (no jam) the time necessary to reach the escape value for the first time gives the breakdown from free flow to congested traffic.

25. **How to Calculate a Traffic Breakdown?** (02.07.2008, R. Mahnke)
Based on Fokker–Planck dynamics an example of initial and boundary value drift–diffusion problem in a finite interval with one reflecting (left) and one absorbing boundary (right) is introduced. After transformation to dimensionless variables, separation ansatz, wave equation, discrete set of wave numbers, superposition, the outflow at right boundary is calculated. Discussion of outflow function (probability per time) at right boundary as well as the cumulative function within an given observation time interval.

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