Vehicular Motion on a Hilly Road

Abstract

We consider the motion of a point-like car on a one-dimensional undulating road under the influence of gravitation and friction. Based on a Newtonian description we investigate the equations of motion of a particle to discuss the speed and position of a car of mass m on a undulating path $f(x)$.

The Model

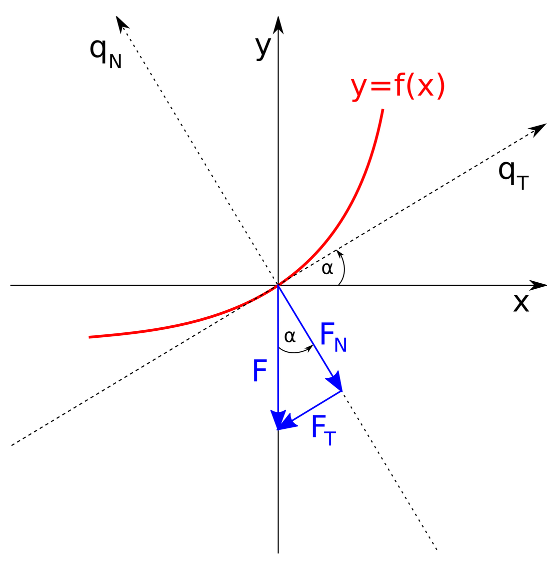

FIGURE 1: In the figure above the vehicular particle follows the path shown as full red line.

Equations of Motion

The motion of a particle of mass m along a one dimensional path of elevation $y = f(x)$ taking both gravitational and frictional forces into account is given by

$$m\,\dot{v}_T = F_T + \gamma\, F_N\, v_T \,, \tag{1}$$
$$\dot{q}_T = v_T \,, \tag{2}$$

where q_T, q_N are the tangential and normal coordinates to the path defined by $f(x)$, v_T and $\dot{v}_T = dv_T/dt$ are the velocity and acceleration in the direction of motion. The gravitational force $F = -mg$ is split into a part that acts in the direction of motion $F_T = F\sin\alpha = -mg\sin\alpha$ and a part that is normal to the path $F_N = F\cos\alpha = -mg\cos\alpha$. The cartesian coordinates x, y and the angle α are defined in the figure above. The coefficient of friction is given as parameter $\gamma \geq 0$.

In cartesian coordinates the equations of motion (1,2) are

$$m\,\dot{v}_x = -mg\frac{f'(x)}{1+f'^2(x)} - m\frac{f'(x)f''(x)}{1+f'^2(x)}v_x^2 - mg\gamma\frac{1}{\sqrt{1+f'^2(x)}}v_x \,, \tag{3}$$
$$\dot{x} = v_x \,, \tag{4}$$

where the first derivative of f is $f'(x) = df(x)/dx$ and $f''(x)$ is the second derivative and $f'^2(x)$ is the square of $f'(x)$. The first and second derivatives with respect to time are denoted by $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$.

The equations (3,4) can be written together as

$$m\,\ddot{x} = -mg\frac{f'(x)}{1+f'^2(x)} - m\frac{f'(x)f''(x)}{1+f'^2(x)}\dot{x}^2 - mg\gamma\frac{1}{\sqrt{1+f'^2(x)}}\,\dot{x} \,. \tag{5}$$

A comparison with the results, reported by S. Geisendorf – see Fig. 2, shows the agreement of only the first terms on the right hand side (although the y-axis is directed down).

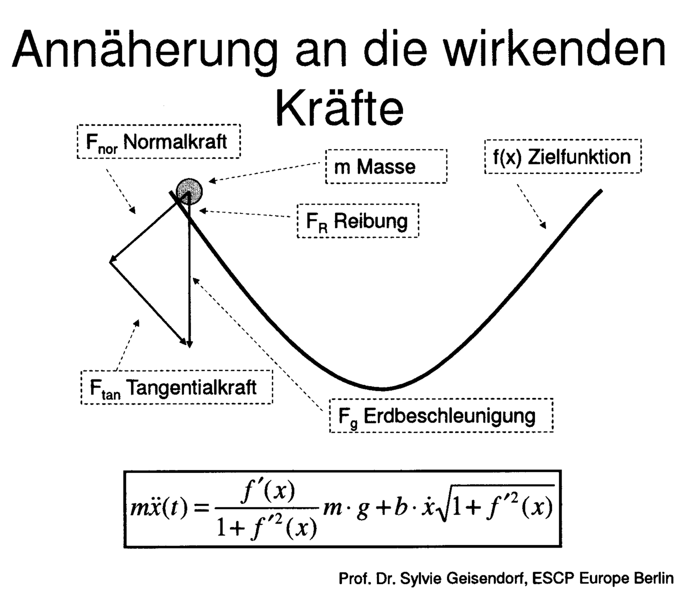

FIGURE 2: From the talk given by Professor Sylvie Geisendorf during the meeting *Physik trifft Volkswirtschaftslehre* at the University of Oldenburg, 21.03.2014.

Special Situations

We present a solution of eq. (5) in two special cases: motion on an arc of a loop road and on a street with a double well.

The motion of a mass on an arc $y = f(x) = -\sqrt{R^2 - x^2}$ with radius R is described by equations

$$m\,\dot{v}_x = -mg\frac{\sqrt{R^2-x^2}}{R^2}x - m\frac{x}{R^2-x^2}v_x^2 - mg\gamma\frac{\sqrt{R^2-x^2}}{R}v_x \,, \tag{6}$$
$$\dot{x} = v_x \,, \tag{7}$$

which in polar coordinates $x = R\sin\varphi$ reads

$$\ddot{\varphi} = -\frac{g}{R}\sin\varphi - g\gamma\,\cos\varphi\,\dot{\varphi} \,. \tag{8}$$

In fact, this is the equation of mathematical pendelum with a new interesting friction term. The solution is illustrated in Fig. 3.

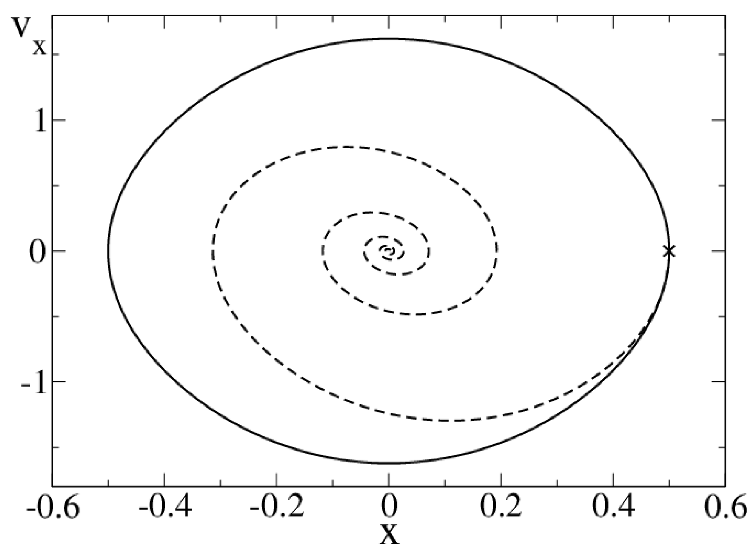

FIGURE 3: Trajectories in the phase space of velocity and coordinate as solutions of (6,7) with parameters $R = 1$ m, $g = 9.81$ m/s^2, $m = 1$ kg for $\gamma = 0$ s/m (thick line) and $\gamma = 0.1$ s/m (dashed line). The initial condition $x(0) = 0.5$ m, $v_x(0) = 0$ m/s is marked by a cross.

As another examle, we consider the motion on a double-well surface

$$f(x) = \frac{a_2}{2}x^2 + \frac{a_3}{3}x^3 + \frac{a_4}{4}x^4 \,, \tag{9}$$

see Fig. 4, with minima x_{min} located at $-(\sqrt{21}-1)/2 \approx -1.79$ und $+(\sqrt{21}+1)/2 \approx 2.79$.

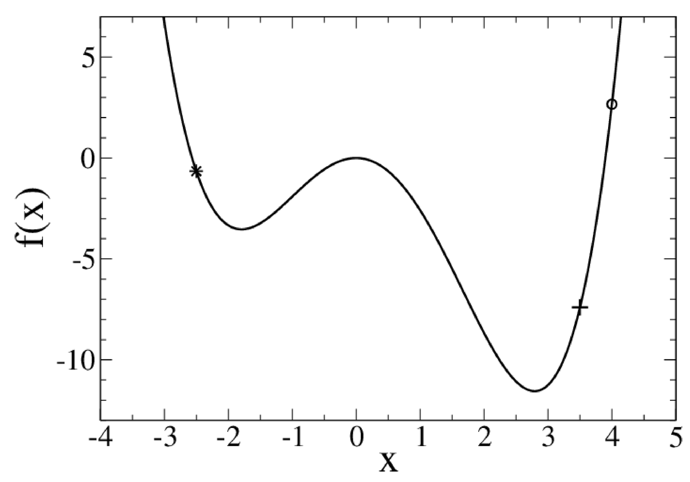

FIGURE 4: The surface (9) with parameters $a_2 = -5$ m^{-1}, $a_3 = -1$ m^{-2} and $a_4 = 1$ m^{-3}. Various initial coordinates are marked by different symbols (cross, plus, asterisk).

The solutions in the phase space of velocity and coordinate is shown in Fig. 5.

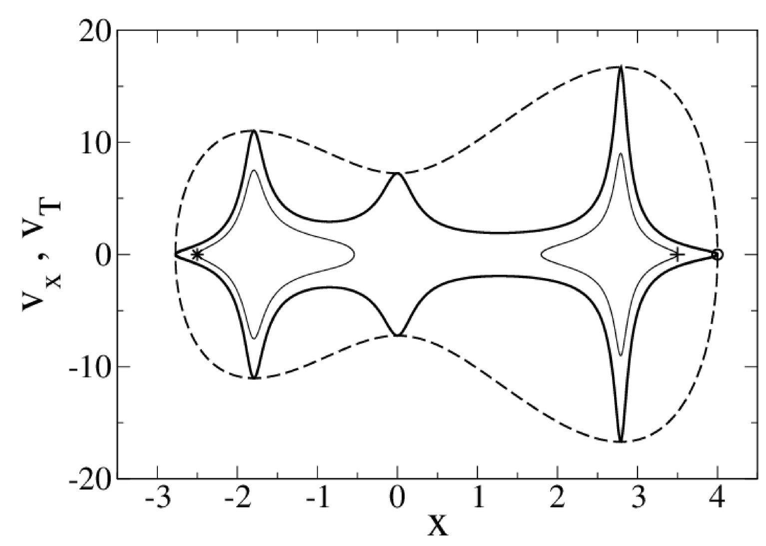

FIGURE 5: Trajectories in the phase space of velocity and coordinate as solutions of (1,2) with parameters $g = 9.81$ m/s^2, $m = 1$ kg and $\gamma = 0$ s/m for $v_x(0) = 0$ m/s and different initial coordinates $x(0)$, shown as in Fig. 4. The dashed line shows the trajectory for v_T at $x(0) = 4$ m, other lines – trajectories for v_x at $x(0) = 4$ m (thick line), $x(0) = 3.5$ m (right thin line) und $x(0) = -2.5$ m (left thin line).

Reinhard Mahnke [a], **Jevgenijs Kaupužs** [b], **Hans Weber** [c]

[a] **Institute of Physics**, Rostock University, Albert-Einstein-Str. 23, D-18051 Rostock, Germany
[b] **Institute of Mathematical Sciences**, University of Liepaja, LV-3401 Liepaja, Latvia
[c] **Department of Physics**, Luleå University of Technology, SE-97187 Luleå, Sweden